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研究課題名（和文）Research on complete quasi-metric spaces with algebraic structure

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研究成果の概要（和文）：本研究では代数的構造を持つquasi-Polish空間とその応用について調べた。Quasi-Polish空間はPolish空間（解析学や測度論でよく応用されている、完備距離づけ可能な可分空間）とω-連続ドメイン（理論的計算機科学でプログラム意味論に応用される位相空間）と第二可算なスペクトル空間（代数幾何学と論理学で応用される位相空間）を含む位相空間のクラスである。本研究では特に半束(semilattice)という代数構造に注目し、ある種のquasi-Polish半束を冪空間モナドのEilenberg-Moore代数として表現できることを示し、そのような半束の位相的な性質や応用について調べた。

研究成果の概要（英文）：This research investigated interactions between algebraic and topological structure on a class of topological spaces called quasi-Polish spaces (i.e., countably based completely quasi-metrizable spaces). Quasi-Polish spaces generalize Polish spaces (which are often used in analysis and measure theory), omega-continuous domains (which are used in theoretical computer science), and countably based spectral spaces (which are used in algebraic geometry and logic). This research mainly focused on algebraic structures known as semilattices, which have important applications in mathematical logic and theoretical computer science.

Our main accomplishments include a careful analysis of several powerspace monads on the category of quasi-Polish spaces which provide a category-theoretical approach (via Eilenberg-Moore algebras) to studying quasi-Polish semilattices. We have also made many new contributions to the general theory of quasi-Polish spaces, which is still a very young research area.

研究分野：mathematical logic, topology, computation

キーワード：quasi-Polish space　topological algebra　semilattices　powerspace

１．研究開始当初の背景

A **metric space** is a set of points with a notion of distance defined between each pair of points. The metric spaces most often encountered in mathematical applications are **complete**, meaning that any Cauchy sequence (i.e., a sequence of points where successive elements become arbitrarily close to each other) will converge to some point in the space. A **Polish space** is a topological space with a countable basis which can be equipped with a complete metric. Most spaces that are used in mathematical analysis, including the natural numbers, the real numbers, the complex numbers, and separable Hilbert spaces, are in fact Polish spaces. The field of **descriptive set theory** investigates the properties and applications of Polish spaces and their definable subspaces.

However, some areas of mathematics and computer science work with topological spaces which cannot be equipped with a metric. For example, theoretical computer scientists often use topological spaces called **ω-continuous domains** to provide semantics for programming languages. Also, the scheme theoretic approach used in modern algebraic geometry involves **spectral spaces** obtained by equipping the prime spectrum of a ring with the Zariski topology. ω-continuous domains and spectral spaces are almost always non-Hausdorff and cannot be equipped with a metric.

In 2013 I introduced a general class of countably based topological spaces called **quasi-Polish spaces**, which includes all Polish spaces, all ω-continuous domains, and all countably based spectral spaces. Quasi-Polish spaces can be defined in a way similar to Polish spaces, but replacing “complete metric” with “complete quasi-metric”. A **quasi-metric** is like a metric, but allows asymmetric distances (so the distance from A to B might be less than the distance from B to A), which makes it possible to equip non-Hausdorff spaces with a quasi-metric. Even though quasi-Polish spaces form a very general class of spaces, they have very nice properties. For example, descriptive set theory naturally generalizes to the entire class of quasi-Polish spaces, and many important topological properties of ω-continuous domains and spectral spaces naturally extend to all quasi-Polish spaces.

２．研究の目的

Although much progress was being made on understanding the topological aspects of quasi-Polish spaces, most mathematical applications involve spaces equipped with additional algebraic structure. The purpose of this research project was to investigate **quasi-Polish spaces with algebraic structure**, and to better understand the interaction between topology and algebra on this class of spaces.

For example, **Polish groups** are Polish spaces equipped with a group structure such that the group operations are continuous functions. Polish groups have a very rich theory, and many applications in fields such as topological dynamics, analysis, and mathematical logic. For example, separable Hilbert spaces are Polish groups with some additional algebraic structure. The completeness properties of Polish spaces make possible some very powerful proof techniques, such as the use of the Baire category theorem.

In theoretical computer science, **semigroups** and **semirings** have many important applications. For example, semigroups are connected with automata theory, and researchers at Paris Diderot used profinite completions of semigroups to successfully apply topological techniques to the study of automata.

Topological semilattices (idempotent commutative semigroups) are also a common theme in domain theory, where they are used in modeling non-deterministic programming languages. Semilattices are also important in mathematical logic and universal algebra.

The goal of this research project was therefore to extend some of the above applications of topological algebra to the more general context of quasi-Polish spaces. By generalizing to quasi-Polish spaces, we hoped that results and methods could be more easily applied between fields. Furthermore, we hoped that generalizing to quasi-Polish spaces would shed new light on the interaction between topological completeness properties and algebraic properties.

３．研究の方法

Every quasi-Polish group is necessarily a regular space, hence Polish. Therefore, one needs to look at generalizations of groups in order to get interesting results for quasi-Polish spaces. In this research, I decided to focus on semigroups (a generalization of a group that possibly lacks an identity element and inverses). Interestingly, I discovered at the end of this research project that Ruiyuan Chen, a PhD student at the California Institute of Technology, independently investigated quasi-Polish groupoids (a different generalization of a group), which allowed him to generalize techniques from descriptive set theory and make important contributions to countable model theory and categorical logic.

The concept of a semigroup is extremely general, so I decided to focus on semilattices (idempotent commutative semigroups). There were several reasons for choosing semilattices: (1) the theory of topological semilattices had been developed and applied in the field of domain theory, and it appeared promising that the theory and applications could be further generalized to quasi-Polish spaces, (2) Reinhold Heckman had recently and independently discovered a duality between quasi-Polish spaces and countably presented frames (an important kind of complete distributive lattice), and it seemed important to further investigate these connections (3) some of the applications of topological semigroups to automata theory had dual lattice theoretical interpretations, so one could expect that a well-established theory of semilattices would also have applications, via duality, to other more general algebraic structures.

I therefore began investigating quasi-Polish spaces whose specialization order was either a join- or meet-semilattice. Andrea Schalk had shown in her PhD thesis that a sober topological join-semilattice is in fact a topological suplattice (a semilattice with arbitrary joins), and an Eilenberg-Moore algebra of the lower powerspace monad on the category of sober spaces. Quasi-Polish spaces are sober, but it was unclear whether the powerspace monads preserved the property of being quasi-Polish. I then began investigating the powerspace monads, and their relationship with semilattices. This research turned out to be deeply connected with ongoing joint work with M. Schroeder and V. Selivanov to classify non-countably based spaces according to the complexity of their open set lattices. It also brought up more connections with locale theory and formal topology, and I began joint work with T. Kawai to start clarifying these connections.

In addition to investigating the relationship between quasi-Polish semi-lattices and the powerspace monads, I continued to develop the general theory of quasi-Polish spaces and their descriptive set theory. This included looking at new characterizations of quasi-Polish spaces, further investigating the topological properties, developing approaches to measure theory and probability theory on quasi-Polish spaces, researching certain subclasses of quasi-Polish spaces that naturally occur in algebraic geometry and theoretical computer science, and investigating categories of spaces that generalize quasi-Polish spaces.

In particular, I worked with M. Schroeder and V. Selivanov on classifying the complexity of QCB0-spaces, which turned out to have interesting connections with the research I was doing on powerspaces. I also worked with A. Pauly on extending descriptive set theory to the category of represented spaces, which further generalizes quasi-Polish spaces. I also worked with A. Pauly on investigating Noetherian quasi-Polish spaces, which naturally arise in applications to algebraic geometry and verification in theoretical computer science.

４．研究成果

During this research, four full-length papers were published, and thirteen presentations were given at international conferences and workshops (this does not include several invited talks given at lab seminars at foreign universities). One additional full-length paper has been submitted to an international journal, but was still under review at the time of writing this report (see the preprint at: https://arxiv.org/abs/1709.06226).

One major achievement of this research was a thorough analysis of the powerspaces of quasi-Polish spaces, and a clarification of their connection with quasi-Polish semilattices and the dual locale theoretic approach to topology.

The first result in this direction was published in the joint paper with M. Schroeder and V. Selivanov, where it was shown that the upper powerspace monad preserves the property of being quasi-Polish. The upper powerspace of a topological space is defined as the set of all (saturated) compact subsets of the space, and is given the upper Vietoris topology. The upper powerspace is a topological meet-semilattice, and for sober spaces it has directed joins, which actually makes it a topological preframe. The observation that the upper powerspace of a quasi-Polish space is still quasi-Polish provided useful information about the structure of the compact subsets of quasi-Polish spaces, and was also used to prove important properties of the open set lattice of quasi-Polish spaces.

Soon afterwards, I proved that the lower powerspace monad also preserves quasi-Polish spaces. The lower powerspace is defined as the set of closed subsets equipped with the lower Vietoris topology. The lower powerspace has the structure of a join-semilattice, and in fact has arbitrary joins, making it a topological suplattice. The preservation proof for the lower powerspace was somewhat different than the proof for the upper powerspace, and revealed important connections with the Baire Category Theorem, which is an important result in descriptive set theory.

These results showed that the upper and lower powerspace constructions were well-defined monads on the category of quasi-Polish spaces. The lower powerspace of a quasi-Polish space gives the free quasi-Polish suplattice over that space. The quasi-Polish Eilenberg-Moore algebras of the lower powerspace are precisely the quasi-Polish join-semilattices (this follows from a characterization given by A. Schalk). Similarly, the upper powerspace of a quasi-Polish space is in some sense a free quasi-Polish preframe over that space, although the precise situation is somewhat more complicated than it is for the lower powerspace. The quasi-Polish Elienberg-Moore algebras of the upper powerspace form a particular class of quasi-Polish meet-semilattices, although a precise characterization of these algebras has not yet been found. In particular, results from A. Schalk show that there exist (locally compact) quasi-Polish meet-semilattices which are not algebras of the upper powerspace. Although I have made much progress on characterizing the quasi-Polish algebras of the upper powerspace monad, a full characterization has not yet been found at the time of this report, and is an important open problem.

I then showed that the upper and lower powerspace monads commute on the category of quasi-Polish spaces, and their composition is equivalent to the double powerspace monad, which is defined as the space of Scott-open subsets of the open set lattice of the space. In fact, I showed that a topological property of quasi-Polish spaces, called consonance, was precisely what was needed for the powerspace monads to commute. Consonance has been investigated by domain theorists and topologists, but in a very different context than the commutativity result shown here.

The above result showed that the double powerspace construction is also a well-defined monad on the category of quasi-Polish spaces. This was somewhat surprising, because the open set lattice of a quasi-Polish space is in general not quasi-Polish, but you return to the category of quasi-Polish spaces by taking the space Scott-open sets of the open set lattice.

The double powerspace of a quasi-Polish space gives (in some sense) the “free” quasi-Polish frame over that space. The quasi-Polish Eilenberg-Moore algebras of the double powerspace monad are topological frames, where the joins are given by the lower powerspace, the meets are given by the upper powerspace, and the distributivity of meets over joins is given by the commutativity of the powerspace monads (this interpretation is made precise in a category theoretical context by Beck distributivity). A full characterization of these quasi-Polish frames was not found, but I did manage to reduce this problem to the problem of characterizing the algebras of the upper powerspace monad.

After presenting my results on the upper, lower, and double powerspace monads at a logic workshop, T. Kawai pointed out some important connections with formal topology and locale theory, in particular with the upper, lower, and double powerlocales. We then began joint work on clarifying and extending these connections. It required a large amount of time to familiarize myself with the locale theoretic approach, but T. Kawai was extremely patient and in the end we obtained some very nice results.

Given a quasi-Polish space X, let A(X) be the lower powerspace, K(X) the upper powerspace, and O(X) the open set lattice with the Scott-topology. T. Kawai and I showed that A, K, and O commute in a “mixed” way on quasi-Polish spaces, namely that O(A(X))=K(O(X)) and O(K(X))=A(O(X)) (where the “=”sign here means homeomorphic). Furthermore, O(X) is an Eilenberg-Moore algebra of both the upper and lower powerspace monads (from which it easily follows that it is also an algebra of the double powerspace monad). This result introduces some interesting dualities that are still being investigated, but it is important because it shows that the algebraic structure of the open set lattice of a quasi-Polish space is completely captured by the powerspace monads. The results also showed that, even though O(X) is not countably based and not quasi-Polish in general, O(X) still has very nice topological properties.

In addition to the above results, I also made progress on developing measure theory for quasi-Polish spaces. In particular, I showed that every locally finite continuous valuation on a quasi-Polish space extends uniquely to a Borel measure, and that the space of continuous valuations on a quasi-Polish space is again quasi-Polish.

I also proved a generalization of an old result by Hurewicz to the context of quasi-Polish spaces, by showing that there are four “canonical” spaces which are (in a precise sense) the simplest examples of non-quasi-Polish spaces. The proof of this result was rather complicated, but recently it has become an important tool in proving characterizations of quasi-Polish spaces.

Also, through joint work with A. Pauly, we proved an interesting characterization of Noetherian quasi-Polish spaces using methods from synthetic descriptive set theory. These spaces have important applications in algebraic geometry and verification in computer science.

５．主な発表論文等
（研究代表者、研究分担者及び連携研究者には下線）

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〔図書〕（計　０件）
該当なし

〔産業財産権〕

○出願状況（計　０件）
該当なし

○取得状況（計　０件）
該当なし

〔その他〕
ホームページ：
https://www.h.kyoto-u.ac.jp/academic_f/faculty_f/142_matthew_d_0/

プレプリント (査読中):
Matthew de Brecht and Tatsuji Kawai. On the commutativity of the powerspace constructions.
https://arxiv.org/abs/1709.06226

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