様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

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研究課題名（和文）ファノ多様体の有界性

研究課題名（英文）Boundedness of Fano varieties

研究代表者
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研究成果の概要（和文）：私の研究の目的は、ファノ多様体の有界性と関連するトピックを研究することです。 極小モデル理論によると、ファノ多様体は双有理幾何学の基本クラスを形成します。 したがって、有界性など、このクラスの基本的な特性を理解することは非常に興味深いです。 族としてのファノ多様体の有界性、そしてファノ多様体のさまざまな不変量の有界性にも興味があります。 私は2つの主な研究成果を上げました。まず、体積が下から制限されたK準安定なファノ多様体が有界族を形成することを証明しました。 第二に、私の共同研究者と、私達はほとんどの一般型三次元多様体に対して最適なネーター不等式を確立しました。

研究成果の学術的意義や社会的意義
K安定性と有界性の両方がファノ多様体の研究の中心的なトピックです。 K安定性はファノ多様体のための良いモジュライ空間を構成するための正しい条件であると期待され、有界性はモジュライの構成に向けた最初のステップです。 K半安定ファノ多様体の有界性に関する私の研究は、これら二つの中心的なトピックを組み合わせたものです。
一方、ネーター不等式は一般型の代数曲面の分類理論において非常に重要です。 我々の結果は、ほとんどの一般型三次元多様体に対して最適なネーター不等式を与え、一般型の三次元多様体の分類理論に役立つことが期待される。 証明で私達はファノ多様体の研究からの多くのアイデアを使用します。

研究成果の概要（英文）：The goal of my research project is to study the boundedness of Fano varieties and related topics. According to Minimal Model Program, Fano varieties form a fundamental class in birational geometry. Hence it is very interesting to understand the basic properties of this class, such as boundedness. This is one of the most important and interesting problems in birational algebraic geometry. I am interested in the boundedness of Fano varieties as a family, and also the boundedness of various invariants of Fano varieties. I want to develop both general theory and explicit calculation. Also I use the methods and results in the study of Fano varieties to study varieties of other type. I made two main research achievements: firstly, I proved that K-semistable Fano varieties with volumes bounded from below form a bounded family; secondly, with my collaborators, we established the optimal Noether inequality for most threefolds of general type.

研究分野：Birational geometry of algebraic varieties

キーワード：Fano varieties　boundedness　K-stability　alpha-invariants　pluri-canonical system　Calabi-Yau varieties　general type　geography problem

様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９、ＣＫ－１９（共通）

１．研究開始当初の背景

Birational geometry is one of the most popular topics in algebraic geometry. I am interested in boundedness of Fano varieties. My starting point is the famous Borisov-Alexeev-Borisov conjecture, which says that, for a fixed positive integer d and a positive real number e, the set of d-dimensional Fano varieties with e-lc singularities forms a bounded family. This conjecture is related to the termination of flips and other topics in birational geometry. At the point of application, I solved two weak versions of this conjecture in dimension 3.

２．研究の目的

The purpose of this research is to investigate the boundedness of Fano varieties, or more general, varieties of Fano type. According to Minimal Model Program, varieties of Fano type form a fundamental class in birational geometry. Hence it is very interesting to understand the basic properties of this class, such as boundedness. This is one of the most important and interesting problems in birational algebraic geometry.

３．研究の方法

I studied Fano varieties from many different aspects:

(1) Singularities: as we are interested in singular Fano varieties, the study of singularities is necessary. Also klt singularities can be viewed as local version of Fano varieties, so the study of singularities interacts with the study of Fano varieties as well. Especially, I study the alpha-invariant which is an important invariant measuring the singularities of Fano varieties and plays important roles in many area such as boundedness and K-stability.

(2) Fano fibrations: in birational geometry, one main idea is to cut down the dimension and do induction. In this process, Fano fibration appears naturally, and the study of Fano fibrations helps us understand more about the geometry of Fano varieties.

(3) Birationality of linear systems: it is important to understand when a linear system gives a birational map, for example, anti-pluri-canonical linear systems of Fano varieties. We have many different ways to study this property, for example, the Angehrn-Siu type argument for general theory, and the Chen-Chen method for explicit geometry of 3-folds.

(4) Surfaces and 3-folds: Other than developing general theory working for any dimension, we can try to firstly solve special cases in lower dimensions. So the study of surfaces and 3-folds help us to understand lower dimensional varieties and sometimes gives clue for general dimensions.

(5) Positivity or boundedness of other invariants: for example, boundedness of volumes, positivity of Chern classes, etc.

４．研究成果

(1) Using the methods and results from boundedness of Fano varieties, with my collaborator, we showed that for a smooth uniruled projective variety and a big and semiample line bundle on it, there exists a proper closed subset containing all subvarieties with bigger fujita a-constant than that of the variety. This confirms a conjecture of Lehmann, Tanimoto, and Tschinkel, and plays an important role in the study of geometry side of Manin's conjecture. This result was published in Algebraic Geometry.

(2) With my collaborator, we investigated torsion exceptional sheaves on a weak del Pezzo surface of degree greater than two whose anticanonical model has at most A-singularities. We show that every torsion exceptional sheaf can be obtained from a line bundle on a (-1)-curve by acting spherical twist functors successively. This result was published in Journal of Algebra.

(3) With my collaborator, we investigated Kawamata's effective non-vanishing conjecture for manifolds with trivial first Chern classes and confirmed it for all hyperkähler varieties of dimension at most 6.

(4) I showed that K-semistable Fano manifolds with the smallest alpha invariant are projective spaces.

(5) I proved that K-semistable Fano varieties with volumes bounded from below form a

bounded family. Both K-stability and boundedness are central topics of the study of Fano varieties. K-stability is expected to be the right condition in order to construct a good moduli space for Fano varieties and boundedness is the first step towards the construction of a moduli. My study of boundedness of K-semistable Fano varieties combines these two central topics. This result was accepted by Annales scientifiques de l'ENS.

(6) With my collaborator, we continued our joint work on the study of the explicit geometry of terminal (weak) Fano 3-folds. We studied the behavior of pluri-canonical system of terminal weak Fano 3-folds, and showed that any terminal weak Fano 3-fold is birational to another terminal weak Fano 3-fold with the 52nd-canonical system giving a birational map. This result generalized our first joint work which is published on J. Differential Geom. in 2016.

(7) Working with collaborators, we used the methods in the study of boundedness of Fano varieties to study boundedness of rationally connected Calabi-Yau varieties, which are varieties with some similar properties with Fano varieties. We showed that rationally connected Calabi-Yau 3-folds with klt singularities form a birationally bounded family, and they form a bounded family modulo flops assuming mld's are bounded away from 1.

(8) Working with collaborators, we considered geography problem for varieties of general type and proved an optimal Noether inequality for 3-folds of general type. Surprisingly, methods used for Fano varieties such as global log canonical thresholds and connectedness lemma are involved and become keys to the final solution of the problem.

５．主な発表論文等

〔雑誌論文〕（計　4 件）

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⑥Chen Jiang, Effective birationality, NCTS Workshop on Singularities, Linear Systems, and Fano Varieties, 2017 年

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⑧Chen Jiang, Birkar's Anti-pluricanonical systems on Fano varieties explanation, Tambara Summer School 2016, 2016 年

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〔図書〕（計　0 件）

〔産業財産権〕
○出願状況（計　　0 件）

名称：
発明者：
権利者：
種類：
番号：
出願年：
国内外の別：

○取得状況（計　　0 件）

名称：
発明者：
権利者：
種類：
番号：
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国内外の別：

〔その他〕
ホームページ等

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６．研究組織

なし