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研究課題名（和文）Perverse sheaves of categories, and derived symmetries of 3-folds

研究課題名（英文）Perverse sheaves of categories, and derived symmetries of 3-folds

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研究成果の概要（和文）：本研究は代数幾何学、すなわち代数方程式の解として決まる空間（解空間）を対象とした。そのような空間は滑らかであるとは限らず、滑らかでない部分は「特異点」と呼ばれる。この分野で重要な問題の一つは、解空間の隠れた対称性を理解することであり、それは通常の対称性を拡張した「導来圏」の対称性を含む。導来圏とは、理論物理における弦理論的方法での空間の研究にも使われる代数的道具である。
第一に、隠れた対称性を「偏屈圏(perverse schober)」という特異的なふるまいを持つ圏の族を用いて調べた。
第二に、変形代数を用いて、ある広いクラスの特異点に新たに隠れた対称性を見つけ、研究を進展させることができた。

研究成果の学術的意義や社会的意義
最初M.Wemyss氏と共同で、特異点に付随する非特異空間に隠れた対称性を見出す一般的な方法を与えた（Adv. Math.に出版済）。これは、以前のこの対称性の研究方法を統一し、特異点の研究への新たな道具の導入となった。次の論文(IMRNに出版済)では、幾何学的不変式論での壁越えと偏屈圏の関連を説明した。偏屈圏を構成する新しい道具を与え、壁越えに潜む隠れた対称性を理解する新しい方法を与えた。最後に桑垣樹氏と共同で、ミラー対称性への応用を与えた。ミラー対称性は、数学と物理に跨る多くの人々によって研究される弦理論的双対性である。特に偏屈圏に対し「ミラー定理」を証明し、この活発な分野の進展に貢献した。

研究成果の概要（英文）：My focus was algebraic geometry, the study of solution spaces to algebraic equations using geometric methods. Such spaces are not necessarily smooth: their non-smooth points are called `singularities'. An important problem is to understand certain hidden symmetries of solution spaces, extending usual symmetries by including symmetries of the `derived category', an algebraic tool which is also related to the string-theoretic study of the space in theoretical physics.
I made progress on this problem in two main ways, completing four research papers with collaborators: first,linking hidden symmetries with `perverse schobers' which give a notion of families of categories with singular behaviour; second, associating them to quite general singularities in algebraic geometry, via deformation algebras.

研究分野：Perverse sheaves and derived symmetries

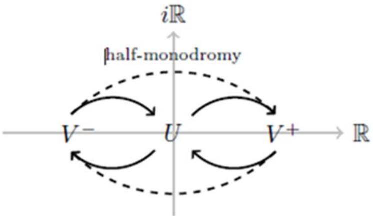
キーワード：Perverse sheaves　Derived symmetries　Mirror symmetry　Flops　Variation of GIT　Deformation theory　Noncommutative algebra

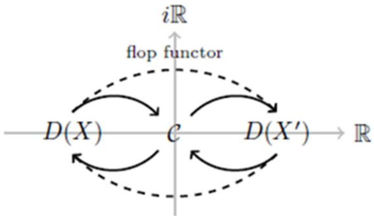
１．研究開始当初の背景

A continuing challenge in algebraic geometry is to understand `derived symmetries' of varieties, that is the symmetries of their derived categories of coherent sheaves. My proposal was to work on two new approaches to derived symmetries: firstly, the `perverse schobers' of Kapranov-Schechtman introduced in 2014, which should be thought of as perverse sheaves of categories; secondly, the approach to constructing and studying derived symmetries using noncommutative deformations which I introduced in 2013, jointly with M. Wemyss.

Perverse sheaves. For a simple example, first note that a local system L on C - 0 may be described by its fibres V± at generic points on the two components of R - 0, along with `half-monodromy' morphisms.

A perverse sheaf on C singular at 0 may be specified by a further U, and morphisms as below, plus conditions. A categorification of this is given by Kapranov-Schechtman, known as a `spherical pair', a prototypical example of a perverse schober. We replace vector spaces with categories, and morphisms with functors: for example, given a 3-fold flop ψ: X → X', we have data as below. The category C here is obtained from the graph of ψ, by work of Bodzenta-Bondal.





Noncommutative deformations. Given smooth 3-folds X and X' related by a flop, Bridgeland constructed certain canonical equivalences F : D(X) $\overset{\sim}{\longleftrightarrow}$ D(X') : F'. These equivalences are not mutually inverse: the theorem below, which is joint work with M. Wemyss, explains this using noncommutative deformations of curves on X.

Let the flopping locus in X have components Ci $\cong$ P1 for i = 1,..., n. At the time of writing my proposal, the following had been proved for the special case n = 1, and I anticipated it was true for general n.

Theorem 1. [5, 6, 7] There exists a C^n-algebra A which represents the functor of noncommutative deformations of the sheaves O_{C_i} (-1) on X. Writing E for the corresponding universal sheaf on X, then:

(1) there is an autoequivalence T_E of D(X), defined as a `twist functor' and

(2) there is a natural isomorphism of functors $\mathsf{T}_E \cong (\mathsf{F}' \circ \mathsf{F})^{-1}$.

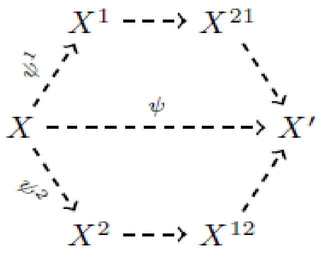
Derived symmetries and hyperplane arrangements. Around the time of application in 2015, I made progress [7] on describing derived symmetries groups of 3-folds in terms of certain generalized braid groups G below. This partly inspired the goals of the application: I give details now.

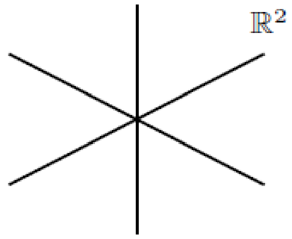
Given a smooth surface S with a crepant contraction to a surface with a simple singularity of Dynkin type Γ, the derived category D(S) carries an action of the braid group of type Γ, by work of Seidel-Thomas and Bridgeland. I proved the following 3-fold analogue, jointly with M. Wemyss.

Theorem 2. [7] For a 3-fold minimal model X admitting a flopping contraction, the derived category D(X) carries an action of $G = \pi_1(\mathbb{C}^n - \mathcal{H}_\mathbb{C})$ for a certain hyperplane arrangement H in R^n. For a first example, let the 3-fold X be a generic 1-parameter deformation of a minimal resolution for a type A_n surface singularity. In this case the hyperplane arrangement H is the type A_n Coxeter arrangement in R^n, and the group G is the pure braid group on n strands. In general, the hyperplane arrangement H is chosen to have the following property.

$$\{\text{connected components of } R^n - H\} \longleftrightarrow \{\text{minimal models } X^i \text{ birational to } X\}$$

In the A_2 example the exceptional locus in X is a nodal curve, with two components. These have individual flops ψ^i and a simultaneous flop ψ. Iterating gives 6 minimal models, as shown below, and derived equivalences between minimal models: the action of Theorem 2 is obtained by composing these.





2．研究の目的

The purpose was to construct schobers for 3-folds, and higher-dimensional varieties, to better understand derived symmetries. In parallel, I planned to deepen the known descriptions of derived symmetries using deformation theory. I explain these in more detail in **Research areas A** and **B** below.

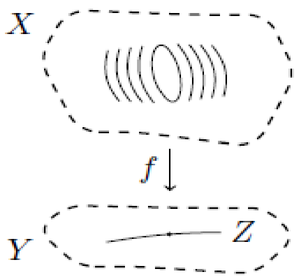
Research area A. I proposed to construct examples of perverse sheaves of 3-fold categories in the setting of Theorem 2 and, in particular, to obtain a perverse sheaf of categories on C^n for each 3-fold minimal model X which, after restricting to C^n - H_C, would give a local system of categories with generic fibre the derived category of X, providing a representation of $\pi_1(C^n$ - $H_C)$ which would recover the derived symmetry action. This would produce a large class of examples of perverse sheaves of categories, and illustrate how they control derived symmetry actions. My plan was to restrict first to flopping contractions of Dynkin type A, where toric methods are applicable. In addition, this would produce a new viewpoint on `flop-flop=twist' theorems, by realizing them as natural relations between `half-monodromies' and `monodromies', respectively, of a schober.

Research area B. I planned to generalize Theorem 1, which I had proved in 2014 for the case n = 1, to the case of general n. In particular, the aim was to show that a derived symmetry for a general 3-fold flopping contraction with flopping locus having components $C^i = P^1$ may be induced by the simultaneous multipointed deformations of the coherent sheaves $\{O_{C^i}\}$ on X. This purpose of this was to deepen the previously-established links between derived symmetries and noncommutative deformations, and to understand more general contractions.

3．研究の方法

Research area A. I began work for toric cases. In the process, I discovered a construction of spherical pairs for a large class of so-called balanced wall-crossings in geometric invariant theory (GIT). Examples include higher-dimensional toric flops over non-isolated singularities. I then worked on extending these results to multi-parameter variations of GIT, to provide tools to construct perverse sheaves of 3-fold categories. I encountered technical problems in defining perverse sheaves of categories on higher-dimensional bases (in particular, questions remain about possible natural conditions on orthogonals to embeddings). At this time Bondal, Kapranov, and Schechtman completed a paper which, in particular, gave a definition of schobers on certain higher-dimensional bases, and studied a key (conjectural) example for the Springer resolution for $\mathfrak{sl}(3)$, corresponding to the A_2 hyperplane arrangement illustrated above. As this work partly satisfied the purpose of my proposed project, I decided to change focus, applying the above GIT wall-crossing results to investigate schobers for higher dimensional varieties, and mirror symmetry for schobers: this is described in the next section.

Research area B. The proposed work in this area was completed ahead of schedule in my paper [7] joint with M. Wemyss. I therefore revised my plan to study more general contractions: I extended our theory of noncommutative deformations to higher dimensions and non-isolated singularities, including a large class of contractions with higher-dimensional fibres, by giving a relative version of the theory. I describe this here, and later its application to derived symmetries.



For a birational contraction f : X → Y satisfying the assumption below, we define a sheaf of algebras A on Y which is supported on the locus Z over which f is not an isomorphism, as pictured.

Theorem 3. [3] Under assumption above, there is a sheaf of algebras A on Y supported on Z, inducing an object E of D(X), such that for points z of Z such that $f^{-1}(z)$ is one-dimensional with components Ci, then:
(1) the completion A_z prorepresents the functor of noncommutative deformations of the sheaves O_{Ci} (-1) on X, up to Morita equivalence;
(2) the restriction of E to the formal fibre of f over z is a sheaf, isomorphic to the universal family corresponding to the prorepresenting object from (1), up to summands of finite sums of sheaves.
I indicate the construction of the sheaf of algebras A, which is a sheafy version of the construction of the algebra A from Theorem 1. Assume a f-relative tilting generator $O_X \oplus N$, and let T denote the relative endomorphism algebra of $O_X \oplus N$, a sheaf of algebras on Y . Then we have

$$T = f_* \mathrm{End}_X(O_X \oplus N) \cong \mathrm{End}_Y f_*(O_X \oplus N).$$

Definition. [3] Let A = T / I, a sheaf of algebras on Y, where I is the ideal of sections of T which factor, at each stalk, through a sum of copies of O_Y .

４．研究成果

Research area A. I describe my construction of schobers associated to GIT quotients, restricting here to the toric case for simplicity.
Take a projective-over-affine variety V with an action of a torus G, and a linearisation M. This determines a semistable locus $V^{ss}(M) \subseteq V$: we take the GIT quotient X to be the quotient stack V^{ss}/G. The complement of V^{ss} has, after certain choices, a GIT stratification by strata S^i, each with an associated one-parameter subgroup λ^i of G: by construction, each S^i contains an open subvariety Z^i of the λ^i-fixed locus.
Take a wall-crossing between two linearizations $\mathbf{M}_\pm$, with a linearization M_0 on the wall. Then the semistable loci for M_ are obtained from the semistable locus for M_0 by removing appropriate strata, say $S^i_\pm$. We use the following notion, specializing a situation considered by Halpern-Leistner.
Definition. [1] A wall-crossing is simple balanced if $V^{ss}(M_0) = V^{ss}(M_\pm)$ $\mathbf{U}\,\mathbf{S}_\pm$ where $\mathbf{S}_\pm$ are strata for linearizations $\mathbf{M}_\pm$, with equal fixed subvarieties Z and inverse one-parameter subgroups $\lambda_- = \lambda_+^{-1}$.
Assume for simplicity V is smooth and G-equivariantly Calabi-Yau, and write GIT quotients $X_\pm$
Theorem 4. [1] Take a simple balanced wall-crossing for a toric GIT problem V/G as above. For each integer w, there exists a spherical pair P determined by semi-orthogonal decompositions

$$< D(X\_) , D(Z/G)^w > = P_0 = < D(X+) , D(Z/G)^{w+\eta} >$$

where superscripts denote λ-weight subcategories, and η denotes the $\lambda_\pm$-weight of $\det \mathcal{N}^\vee_{S_\pm} V$ *on* Z.

Following this work, I published a paper [4] applying Theorem 4 to construct perverse sheaves of categories on the complex plane, thought of as the complexification of a one-parameter GIT stability space, as follows.
Theorem 5. [4] For a standard flopping contraction of X with exceptional locus $E \cong \mathbb{P}^n$, there exists a schober on C, singular at iZ, with
- generic fibre D(X/E), the category of objects in D(X) set-theoretically supported on E, and
- monodromy around iw $\in$ iZ given by the spherical twist of $O_E(w)$.

The complex plane C above can be related to a chart on a Riemann surface whose non-trivial topology reflects relations between derived symmetries of X, in a picture outlined by Kapranov-Schechtman. In further results in [4], I worked this out in detail for standard flops, including flops of orbifold projective spaces. For the Atiyah flop case, I also constructed a schober on a (partial compactification) of a stringy Kaehler moduli space: this idea has interesting mirror symmetry applications, as follows.
In collaboration with Tatsuki Kuwagaki [4], I proved mirror theorems for schobers, and applied them to give new proofs of mirror symmetry for certain associated singularities. In particular, we proved the following.
Theorem 6. [4] A schober as in Theorem 5, for the case of the Atiyah 3-fold flop, has a mirror partner given by a certain diagram of Fukaya categories with stops.
This mirror theorem was obtained using previous work of Ganatra-Pardon-Shende relating

constructible sheaf categories to Fukaya categories with stops, and of Kuwagaki on the coherent-constructible correspondence: the starting point for our work was the observation that, for the Atiyah flop, the existing schober construction had a natural counterpart under the coherent-constructible correspondence. Bondal, Kapranov, and Schechtman had previously explained how to take cohomology of certain schobers: we showed that in our cases this yielded a new proof of mirror symmetry for the 3-fold conifold singularity, as a corollary of Theorem 6.

Research areas B. My work [3] extended the principle of constructing derived symmetries using noncommutative deformations, to include higher-dimensional varieties and non-isolated singularities, by giving a relative version of the theory which produces a sheaf of deformation algebras. The following theorem gives associated derived symmetries under certain natural assumptions.

Theorem 7. [3] In the setting of Theorem 3, and with crepant f, and Y complete locally a hypersurface at each point of Z, and either

(i) codimZ $\geq$ 3, or

(ii) the sheaf A is Cohen-Macaulay, and E is perfect,

then there is a Fourier-Mukai autoequivalence T_E of D(X), fitting into a distinguished triangle of functors

$$f^{-1}\mathbb{R}f_*\mathbb{R}\mathcal{H}om_X(\mathcal{E},-)\underset{f^{-1}\mathcal{A}}{\overset{\mathbb{L}}{\otimes}}\mathcal{E} \longrightarrow \mathrm{Id}_{D(X)} \longrightarrow \mathrm{T}_{\mathcal{E}} \longrightarrow .$$

Remark. The construction of T_E here generalizes the construction of T_E from Theorem 1, to which it reduces when Z is a point. A key example where the theorem applies to a contraction with higher-dimensional fibres is the Springer resolution of the variety of singular d-by-d matrices. In this example and many others, it is an interesting subject of further study to relate T_E to known autoequivalences associated to contractions.

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〔図書〕（計 0 件）
〔産業財産権〕
○出願状況（計 0 件）
○取得状況（計 0 件）
〔その他〕

６．研究組織

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