様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

科学研究費助成事業　　研究成果報告書

令和　元 年　 ５ 月 ２７ 日現在

機関番号：１１３０１
研究種目：若手研究(B)
研究期間：2016～2018
課題番号：１６Ｋ１７６１６

研究課題名（和文）Spectral measures of random matrices and universality of random Jacobi matrices

研究課題名（英文）Spectral measures of random matrices and universality of random Jacobi matrices

研究代表者
Ｔｒｉｎｈ　Ｋｈａｎｈ・Ｄｕｙ（Trinh, Khanh Duy）

東北大学・数理科学連携研究センター・准教授

研究者番号：００７２６１２７

交付決定額（研究期間全体）：（直接経費）　　2,700,000円

研究成果の概要（和文）：結合確率密度関数による直交/ユニタリ/斜交アンサンブルの自然な一般化ガウシアンベータアンサンブルは、独立成分を持つヤコビ行列と呼ばれる対称三重対角行列の固有値として実現されることが知られている．本研究では、ガウシアンベータアンサンブルについて、スペクトル測度・経験分布の収束やその極限近傍でのガウス的揺らぎなど、いくつかの新たなスペクトル特性を確立する．その研究手法は主にランダム行列モデルに基づいているが、広いクラスのランダムヤコビ行列にも適用可能である．

研究成果の学術的意義や社会的意義
This research establishes several new spectral properties for a random matrix model so called Gaussian beta ensembles. Approaches are also new and are applicable to a large class of random Jacobi matrices.

研究成果の概要（英文）：Gaussian beta ensembles, a natural generalization of Gaussian orthogonal/unitary/symplectic in terms of the joint probability density functions, are now realized as eigenvalues of random symmetric tridiagonal matrices, called Jacobi matrices, with independent entries. In this research, we establish several new spectral properties of Gaussian beta ensembles such as convergence to a limit and Gaussian fluctuations around the limit of the spectral measures and of the empirical distributions. Approaches which are mainly based on the random matrix model are also applicable to a large class of random Jacobi matrices.

研究分野：Probability theory, Random matrix theory

キーワード：Gaussian beta ensembles　radom Jacobi matrices　spectral measures　semi-circle law

１．研究開始当初の背景

Gaussian beta ensembles were originally defined as ensembles of real particles with explicit joint probability density functions which generalize Gaussian orthogonal/unitary/symplectic ensembles. They can be viewed as the equilibrium measure of a one-dimensional Coulomb log-gas under a Gaussian potential at the inverse temperature beta. A random symmetric tridiagonal matrix (called a Jacobi matrix) model for them was introduced by Dumitriu and Edelman in 2002. Since then, many new spectral properties such as the limiting behaviors of the local statistics and of the largest eigenvalue have been derived by using the model. Universality phenomenon, meaning that a property which holds for Gaussian beta ensembles also holds for a large class of beta ensembles or a large class of random Jacobi matrices, has also been studied intensively.

２．研究の目的

The purpose of this research is to establish new spectral properties of Gaussian beta ensembles and of random Jacobi matrices. This research aims to provide universal approaches which are applicable to a large class of random matrices.

３．研究の方法

Since Gaussian beta ensembles and random Jacobi matrices are related to many fields of mathematics and physics, this research has been done by exchanging ideas with people from different fields through attending seminars, workshops and conferences.

４．研究成果

The following are some main achievements.

(1) Spectral measures of random Jacobi matrices. Three classical beta ensembles on the real line are now realized as eigenvalues of certain random Jacobi matrices. The limiting behavior of the empirical distributions has been well studied. In this research, we study the limiting behavior of the spectral measures. It was motivated by the fact that a finite Jacobi matrix of size N is one to one correspondence with a probability measure, called its spectral measure, supported on exactly N real points. A universal approach is developed to show the convergence to a limit and Gaussian fluctuations around the limit of the spectral measures of random Jacobi matrices of the three classical beta ensembles. Furthermore, for these three beta ensembles, relations between their spectral measures and empirical distributions can be made explicitly. Thus, studying spectral measures provides another way to establish the limiting behavior of empirical distributions.

(2) Global spectrum properties of Gaussian beta ensembles. For fixed beta, the empirical distributions of Gaussian beta ensembles converge to the semi-circle distribution, known as Wigner's semi-circle law. What happens if the parameter beta varies with the matrix size? This study gives the answer. We show that:(i) Wigner's semi-circle law holds as long as $N\beta$ tends to infinity, and (ii) in case $N\beta$ goes to a finite constant, the empirical distributions converge to a probability measure with full support which is the probability measure of associated Hermite orthogonal polynomials. Gaussian fluctuations around the limit, or more precisely, central limit theorems for linear statistics are also established for test functions having continuous derivative of polynomial growth.

(3) Determinants of Gaussian beta ensembles. The determinant of a matrix is an important quantity. For fixed beta, a central limit theorem for the log-determinants of Gaussian beta ensembles is established. Note that the log-determinant can be written as the linear statistics with respect to the logarithm test function. However, the logarithm does not belong to the class of test functions mentioned in Problem (2) above. Thus, it should be dealt with separately. Note also that the absolute value of determinants of Gaussian orthogonal ensembles and of Gaussian unitary ensembles can be written as products of independent random variables, implying a central limit theorem by using some classical limit theorem in probability theory.

(4) Poisson statistics for the local statistics of Gaussian beta ensembles in the regime where $N\beta$ stays bounded. In the regime where $N\beta$ tends to a finite constant, the Poisson statistics was established previously by analyzing the joint density of

Gaussian beta ensembles but without explicit form of the intensity measure. We give an alternative proof based on the random Jacobi matrix model. We first refine Minami's method to show the Poisson statistics for general Jacobi matrices with independent entries under some mild conditions. Then we show that such conditions hold true in case of Gaussian beta ensembles in the considering regime.

５．主な発表論文等

〔雑誌論文〕（計　4 件）

[1] F. Nakano and K.D. Trinh, Gaussian beta ensembles at high temperature: eigenvalue fluctuations and bulk statistics, *J. Stat. Phys.* **172**(2) (2018), 295–321
DOI: https://doi.org/10.1007/s10955-018-2131-9

[2] K.D. Trinh, Global spectrum fluctuations for Gaussian beta ensembles: a martingale approach, *J. Theor. Probab.* (Online First)
DOI: https://doi.org/10.1007/s10959-017-0794-9

[3] K.D. Trinh, On spectral measures of random Jacobi matrices, *Osaka J. Math.* **55** (2018), no. 4, 595–617

[4] K.D. Trinh, Distributions of the determinants of Gaussian beta ensembles, 数理解析研究所講究録 (2017), 2023: 77–85 (Non-refereed)

〔学会発表〕（計　12 件）

[1] K.D. Trinh, Gaussian beta ensembles in global regime, MSJ Spring Meeting 2019, 2019

[2] K.D. Trinh, Some aspects of beta ensembles at high temperature, Spectra of Random Operators and Related Topics, 2019

[3] K.D. Trinh, Classical beta ensembles in global regime, 第 165 回学習院大学スペクトル理論セミナー, 2018

[4] K.D. Trinh, Classical beta ensembles at high temperature, Random matrices and their applications, 2018

[5] K.D. Trinh, Global spectral properties of Gaussian beta ensembles, 大阪大学確率論セミナ, 2018

[6] K.D. Trinh, On infinite Jacobi matrices related to beta ensembles, The 16th annual Korea-Japan Workshop on Algebra and Combinatorics, 2018

[7] K.D. Trinh, On associated orthogonal polynomials, Workshop on "Random matrices, determinantal processes and their related topics", 2017

[8] K.D. Trinh, Poisson statistics for Gaussian beta ensembles at high temperature, 2017 年度確率論シンポジウム, 2017

[9] K.D. Trinh, On i.i.d. Jacobi matrices and Gaussian beta ensembles at high temperature, 第 152 回学習院大学スペクトル理論セミナー, 2016

[10] K.D. Trinh, Global asymptotic behaviours of Gaussian beta ensembles, First Kyushu-UNSW Joint Workshop on the Mathematics underpinning Industry and Innovation, 2016

[11] K.D. Trinh, On local statistics of random Jacobi matrices, Spectra of Random Operators and Related Topics, 2016

[12] K.D. Trinh, Some results on Gaussian beta ensembles at high temperature, XII Brunel-Bielefeld Workshop on Random Matrix Theory, 2016

〔図書〕（計　0 件）

〔産業財産権〕
○出願状況（計　0 件）

○取得状況（計　0 件）

〔その他〕
ホームページ等

６．研究組織

(1)研究分担者
該当者なし

(2)研究協力者
研究協力者氏名：中野　史彦
ローマ字氏名：　Nakano Fumihiko