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研究課題名（和文）Cut locus and variational problems with constaints on Finsler manifolds

研究課題名（英文）Cut locus and variational problems with constaints on Finsler manifolds

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研究成果の概要（和文）：本研究で、フィンスラー多様体における最小跡(cut locus)及び変分問題を研究しました。これらは、近代数学の基本的な分野の一つである微分幾何学の重要な課題です。
フィンスラー多様体というのは、物理的かつ幾何学的な性質が方向によって異なるという空間です。すなわち、二点間の距離、最短道などの幾何学はその方向によって違います。日常的なユークリッド空間であれば、点Aから点Bまでの距離は点Bから点Aまでの距離に等しいですが、強い風が吹いている状況を想像しましょう。一定速度で移動する場合、二点間に移動する時間を距離と見なせます。その時に明らかに追い風の場合や向かい風の場合の移動時間が異なります。

研究成果の学術的意義や社会的意義
本研究で、フィンスラー多様体の最小跡に関する新しい結果を得ることができました。最小跡というのは、距離関数は滑らかではなく、特異点を持っているところです。このような点を超えると、今まで最短だった道は最短性を失ってしまうので、非常に重要です。そのために、制御理論、様々な科学の分野、産業などにたくさん応用があります。
また、変分問題やフィンスラー多様体の幾何学との間の関係を明確にし、それを量子力学の分野に応用しました。これらの成果は数学や物理学の国際誌の投稿し、一部はすでに受理され、閲覧可能になっています。

研究成果の概要（英文）：In this research I have studied the cut locus and variational problems on Finsler manifolds. Both of them are important problems in Differential Geometry, one of the most old and fundamental fields of modern mathematics.
Finsler manifolds are spaces where the geometry and Physics of the space depend on the direction. Therefore, distances between points, shortest paths (called geodesics) and many other geometrical properties depend on the direction. In the Euclidean space, distances between points are the same in one direction as well as in the oposite direction, however just imagine a strong wind blowing. If traveling with a constant speed engine, clearly we can regard the time needed to reach point B from a point A as the distance between the points A and B. The everyday life experience teaches that the time needed to travel from A to B is not the same as going back under a strong wind. Finsler geometry is a realistic model of the real World and this make it very important.

研究分野：微分幾何学

キーワード：Finsler manifolds　Riemannian manifods　geodesics　cut locus　variational problem　Quantum mechanics

1 Initial background of the research

Finsler Geometry is just the Riemannian Geometry without the quadratic restriction (S. S. Chern). Indeed, what we call today a **Finsler norm** was actually introduced by B. Riemann in his famous Habilitation Dissertation from 1854, namely a metric function $ds = F(x^1, \ldots, x^n; dx^1, \ldots, dx^n)$ that depends on position and direction.

Any geodesic γ emanating from a point p in a compact Finsler manifold loses the global minimising property at a point q on γ. Such a point q is called a cut point of p along γ. The cut locus of a point p is the set of all cut points along geodesics emanating from p. The cut locus often appears as an obstacle when we try to prove some global theorems in differential geometry.

The cut locus is also a vital notion in analysis, where it appears as a singular points set. The structure of the cut locus plays an important role in optimal control problems in space and quantum dynamics allowing to obtain global optimal results in orbital transfer and for Lindblad equations in quantum control.

2 The purpose of the research

1. The present research proposal has the main purpose to study the geometry and topology of Finsler manifolds by using the properties of distance function and the cut locus.

2. The present research proposal uses the theory of geodesics on Riemannian and Finsler manifolds. This theory is intimately related to the Calculus of Variations.

3 Methods

1. We will use the theory of geodesics and in special the Lipschitz continuity of the distance function, the structure theorems of the cut locus and the analytical properties of Busemann functions on complete, non-compact Riemannian and Finsler manifolds for obtaining new relations of Finsler metrics with the topology of the manifold.

2. We will use the variational problem for integral manifolds (1 dimensional and higher dimensional) of exterior differential systems. This

method will allow to study new geometrical problems not only in Finsler, but also in Riemannian geometry.

4 Results

4.1 The geometry and topology of Finsler manifolds by using the properties of distance function and the cut locus

1. **The geodesics behavior and cut locus of some Finsler manifolds** ([1]).

   We studied the geometry of geodesics of a Kropina space with the fundamental metric obtained by the Zermelo's navigation problem with navigation data given by a unit Killing vector field W on a Riemannian manifold (M, h), called *a strong Kropina space*.

   Theorem 4.1 *(a) Any compact connected manifold with boundary admits a globally defined Kropina metric.*

   (b) Any compact connected boundaryless manifold M admits a globally defined Kropina metric if and only if $\chi(M) = 0$.

   (c) Any compact connected odd dimensional manifold (regardless it has boundary or not) admits a globally defined Kropina metric.

   (d) Any connected non-compact manifold admits a globally defined Kropina metric.

   (e) If M admits a globally defined Kropina metric, then the product manifold $M \times N$ also admits a globally defined Kropina metric.

   (f) Every Lie group G admits $n = \dim G$ distinct globally defined Kropina metrics, one of each corresponding to one vector field in a parallelization of G.

   (g) If M and N are parallelizable, then $M \times N$ admits $m \times n$ distinct globally defined Kropina metrics, where m and n are the dimensions of M and N, respectively.

   Theorem 4.2 *Let M be a 2-dimensional manifold admitting a strong Kropina metric induced by the navigation data (h, W). Then* *(i) The Riemannian surface (M,h) is flat, i.e. M is isometric to one of the manifolds: Euclidean plane, straight cylinder (in the non-compact case), or flat torus, Möbius band, Klein bottle (in the compact case).*

(ii) In the cases M isometric to Möbius band, Klein bottle, W is quasi-regular.

A Hopf-Rinow Theorem for strong Kropina spaces can be now formulated. We also have performed a detailed research of the conjugate locus, cut locus of a such Finsler metric illustrated several with examples.

2. **The cut locus of a Randers rotational 2-sphere of revolution** ([2]).

   Formally, if we consider the background landscape to be a Riemannian manifold (M,h), endowed with a vector field W on M, $\|W\|_h < 1$, then the shortest time travel paths are precisely the geodesics of a Finsler metric of Randers type $F(x,y) = \alpha(x,y) + \beta(x,y) = \frac{\sqrt{\lambda \cdot \|y\|_h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}$ uniquely induced by the navigation data (h,W). Here $W = W^i \cdot \frac{\partial}{\partial x^i}$ is the velocity vector field of the wind, $\lambda = 1 - \|W\|_h^2$, $W_0 = h(W,y)$.

   Theorem 4.3 *Let (M,F) be a Randers rotational 2-sphere of revolution with navigation data (h,W), where $W = \mu \cdot \frac{\partial}{\partial \theta}$ is the wind blowing along parallels, $\mu < \{\frac{1}{\max\{m(r)\}} : r \in [0,2a]\}$, with a pair of poles p,q, $d_h(p,q) = 2a$ such that M is symmetric with respect to $\{r = a\}$, and the flag curvature $\mathcal{K}$ is monotone along a meridian. Then the F-cut locus $\mathcal{C}_x^F$ of a point $x \in M \setminus \{p,q\}$ with $\{\theta(x) = 0\}$ is*

   (a) The subarc of the opposite half bending meridian,

   $$\mathcal{C}_x^F = \varphi(d(x,\tau(t)),\tau(t)), \quad t \in [c, 2a-c],$$

   where φ is the flow of the wind, when $\mathcal{K}$ is monotone non-increasing.

   (b) The following subarc of the antipodal parallel $\{r = 2a - r(x)\}$ to x:

   $$\mathcal{C}_x^F = r^{-1}(2a - r(x)) \cap \theta^{-1}\{\mathcal{H}(m) + \psi(x), 2\pi - (\mathcal{H}(m) - \psi(x))\}.$$

   where $\psi(x) = \mu \cdot d_h(x,\hat{q}_0)$, $\hat{q}_0$ is the h-first conjugate point of x with respect to h, $m := m(r(x))$, when $\mathcal{K}$ is monotone non-decreasing.

(c) *A single point on the antipodal parallel* $\mathcal{C}_x^F = (2a - r(x), \pi(1+\mu R))$, *where* R *is radius of sphere, when* $\mathcal{K} = \frac{1}{R^2}$ *is constant.*

(d) *If the cut locus of* $x \in M \setminus \{p, q\}$ *is a single point, then* $\mathcal{K}$ *is constant.*

More generally, if the Gaussian curvature of h, or of F, is not monotone, we gave a characterization of the cut locus in this case also.

4.2 Study of variational problems for integral manifolds of EDS

1. **Variational problem for time-depending Lagrangians** ([3])

   An an application of our research on Variational Problems, we were interested here to find a Finsler type geometrization of quantum mechanics for charged particle under the influence of an exterior electromagnetic field.

   Theorem 4.4 *The fundamental function* $F = \frac{\alpha^2}{\beta}$, *associated to the Lagrangian of the hydrodynamic representation of the quantum mechanics, in the presence of external electromagnetic fields, is a globally defined Kropina metric on the extended configurations space* $\mathcal{M}$, *where* $\alpha^2 = a_{IJ} y^I y^J$ *is the associated Riemannian metric to the hydrodynamic Lagrangian of the quantum mechanics, and* $\beta = y^0$.

References

[1] Sorin V. Sabau, K. Shibuya, R. Yoshikawa, Geodesics on strong Kropina manifolds, European Journal of Mathematics (2017), 3:1172-1224.

[2] R. Hama, K. Jaipong, Sorin V. Sabau, The cut locus of a Randers rotational 2-sphere of revolution, Publ. Math. Debrecen, 93/3-4 (2018), 387-412.

[3] with S. Liang, Sorin V. Sabau, T. Harko, Finslerian Geometrization of Quantum hydrodynamics in the presence of electromagnetic fields, Physical Reviews D 100, 105012-1 to 105012-15 (2019).

５．主な発表論文等

〔雑誌論文〕　計3件（うち査読付論文　2件／うち国際共著　2件／うちオープンアクセス　1件）

１．著者名 R. Hama, K. Jaipong, S. V. Sabau	４．巻 93/3-4
２．論文標題 The cut locus of a Randers rotational 2-sphere of revolution	５．発行年 2018年
３．雑誌名 Publicationes Mathematicae Debrecen	６．最初と最後の頁 387-412
掲載論文のDOI（デジタルオブジェクト識別子） 10.5486/pmd.2018.8126	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Sabau Sorin V.、Shibuya Kazuhiro、Yoshikawa Ryozo	４．巻 3
２．論文標題 Geodesics on strong Kropina manifolds	５．発行年 2017年
３．雑誌名 European Journal of Mathematics	６．最初と最後の頁 1172～1224
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s40879-017-0189-6	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

１．著者名 S. Liang, Sorin V. Sabau, T. Harko	４．巻 100
２．論文標題 Finslerian Geometrization of Quantum hydrodynamics in the presence of electromagnetic fields,	５．発行年 2019年
３．雑誌名 Physical Reviews D	６．最初と最後の頁 1-105012
掲載論文のDOI（デジタルオブジェクト識別子） 10.1103/physrevd.100.105012	査読の有無 無
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

〔学会発表〕　計3件（うち招待講演　1件／うち国際学会　2件）

１．発表者名 Sorin V. Sabau
２．発表標題 The Geometry of a positively curved Zoll surface of revolution
３．学会等名 日本数学会　春学会
４．発表年 2018年～2019年

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２．発表標題 Finsler metrics on sphere of constant flag curvature
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１．発表者名 Sorin V. Sabau
２．発表標題 Remarks on the Geometry of complete non-compact Finsler manifolds
３．学会等名 International conference in applied and pure mathematics ICAPM 2017（招待講演）（国際学会）
４．発表年 2017年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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６．研究組織

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