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研究課題名（和文）On the algebra and combinatorics of hyperplane arrangements

研究課題名（英文）On the algebra and combinatorics of hyperplane arrangements

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研究成果の概要（和文）：16編のアクセプトされた論文がある。共同研究者と共に（超平面配置の）自由性に関する新しい特徴付けを見出した。この際、我々はCoCoA仕様の計算パッケージを制作し、プラスワン生成超平面配置、非アルティンの場合でのk-レフシェッツ性質などについて研究した。別の共同研究者とはグラフの支配と多重全支配集合について研究し、単純グラフに関連する中間グラフの支配、全支配、全支配者の彩色数について記述した。

研究成果の学術的意義や社会的意義

My research advanced our current knowledge on hyperplane arrangements, in particular on the class of free ones, and our knowledge on domination problems for graphs. In addition, my research connected the study of the Lefschetz properties and of the Jacobian algebra of arrangements.

研究成果の概要（英文）：I obtained 16 accepted publications. With Bigatti and Palezzato, we found new characterizations of freeness. With Palezzato, we determined when an arrangement and its reduction modulo a prime have isomorphic lattices; we wrote a package for CoCoA to do computations with arrangements; we studied free and plus-one generated arrangements, we described how to compute the associated prime ideals of their Jacobian ideal and we proved that the localization of a plus-one generated arrangement is free or plus-one generated. We studied the k-Lefschetz properties in the non-Artinian case. I studied the relations between freeness over finite fields and the rationals for multiarrangements. With Pahlavsay and Palezzato, we studied domination and k-tuple total domination sets of graphs. With them and Kazemnejad, we described the domination, the total domination and the total dominator coloring numbers of the middle graph associated to a simple graph.

研究分野：Algebra

キーワード：Hyperplane arrangements Lefschetz property Graph theory Freeness

1. 研究開始当初の背景

Arrangements of hyperplanes:

An *arrangement of hyperplanes* is a finite collection of codimension one affine subspaces in a finite dimensional vector space. Associated to these spaces, there is a long list of algebraic, combinatorial and topological invariants. Arrangements are easily defined but they lead to deep and beautiful results that put in connection various area of mathematics. We refer to [OT92] for a comprehensive account of this subject. One of the main goals in the study of arrangements is to decide if a given invariant is combinatorial, and, if so, to express it explicitly in terms of the intersection lattice of the arrangement.

In the theory of hyperplane arrangements, the *freeness* is a very important algebraic property. In fact, freeness implies several interesting geometric and combinatorial properties of the arrangement itself, see [OT92]. By definition, an arrangement is free if and only if its module of logarithmic derivations is a free module. A lot it is known about free arrangements, however there is still some mystery around the notion of freeness. For example, Terao's conjecture asserting the dependence of freeness only on the combinatorics is the longstanding open problem in this area. The notion of freeness was introduced by Saito in [Sa80] for the case of hypersurfaces in the analytic category. The special case of hyperplane arrangements was firstly studied by Terao in [Te80], where he showed that we can pass from analytic to algebraic considerations.

Domination problems in graph theory:

Given a simple graph G , we can define several "domination sets". In my research, I focused on the following three. A *dominating set* of a G is a subset S of the vertex set such that every vertex of G not in S is adjacent to a vertex in S . The *domination number* of G is the minimum cardinality of a total dominating set of G . A *total dominating set* of a G is a subset S of the vertex set such that every vertex of G is adjacent to a vertex in S . The *total domination number* of G is the minimum cardinality of a total dominating set of G . A *total dominator coloring* of a graph G is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some color class. The *total dominator chromatic number* of a graph is the minimum number of color classes in a total dominator coloring.

In addition, given a simple graph G , we can construct several "new graphs" from it. The one central to my research is the notion of *middle graph*. Given G , the middle graph of G is the graph $M(G)$ whose vertex set is the union of the vertex set $V(G)$ and edge set $E(G)$ of G , and two vertices x, y of $M(G)$ are adjacent if x, y are in $E(G)$ and x, y are adjacent in G , or x is in $V(G)$ and y in $E(G)$, and x, y are incident in G .

2. 研究の目的

I am dedicated to research in pure mathematics, in particular in the areas of algebra, combinatorics and algebraic geometry. The main purpose of my research is to study fundamental combinatorial, algebraic and topological properties of hyperplane arrangements and to understand the relations

between these properties. Moreover, I am also interested in domination problems for simple graphs. In my career so far, I have been interested in the study of:

- free divisors, with particular attention to free hyperplane arrangements, and the connected Terao's conjecture;
- combinatorial, matroidal and modular aspects of hyperplane arrangements;
- simple, signed and gain graphs along with their connection to hyperplane arrangements;
- multi-nets, local systems, characteristic and resonant varieties of hyperplane arrangements together with the related central lower series and Falk invariants;
- domination problems in graph theory;
- k -Lefschetz properties, generic initial ideal, sectional matrices and Hilbert series in commutative algebra;
- Groebner basis and algorithms from the computer algebra point of view.

The main goal of this project was to advance our current knowledge on hyperplane arrangements, and in particular on the class of free ones, in such a way to get closer to prove Terao's conjecture and connect other areas of mathematics to the study of arrangements. In addition, the project boosted our understanding and knowledge of domination problems for middle graphs.

3. 研究の方法

This research project extended and unified my previous works on arrangements and related topics, and the studied questions arose naturally while working on my past projects and collaborations. A major part of my previous research was concerned with various aspects of the algebra and combinatorics of hyperplane arrangements, and with the study of several invariants of graphs. In particular, I studied in depth the notion of freeness and supersolvability, the connection between arrangements and various types of graphs. This project was concerned exactly with these objects and builded naturally on my collaborations with Bigatti, Yoshinaga, Palezzato, Guo, Suyama and Tsujie.

In each of the publications I wrote during this project, I used different techniques coming from my previous works and my undergraduate and graduate studies.

For [PT21], we used the work we did in [PTfree20] and our knowledge of Groebner basis and their computations over principal ideal domains.

For [PTloc21], we used the work we did in [BPT20] and our knowledge of resolutions of modules over polynomial rings.

For [PT20] and [PTklef21], we used the work we did in [BPT18] and [BPT20] and our knowledge commutative sheaf theory.

For [Tor22], I extended and used the work of [PTfree20], and I also used various ideas coming from module theory over polynomial rings.

For [PPT21], [KPPT22], [KPPTtot23] and [KPPTdomin23], we used ideas from [PPT22] and my previous studied of graph theory.

4. 研究成果

From the start of the grant, I obtained several results on hyperplane arrangements.

In [PT21], with Palezzato, we study the combinatorics of hyperplane arrangements over arbitrary fields. Specifically, we determine in which situation an arrangement and its reduction modulo a prime number have isomorphic lattices via the use of minimal strong Groebner bases applied to the Jacobian ideal of the arrangement. Moreover, we prove that the Terao's conjecture over finite fields implies the conjecture over the rationals.

In [PTloc21], with Palezzato, we continued our studies on free arrangement and we investigated the notion of plus-one generated arrangements introduced in [Abe19]. In particular, we described how to compute the associated prime ideals of the Jacobian ideal of such an arrangement from its lattice of intersection and we proved that the localization of a plus-one generated arrangement is free or plus-one generated.

In [PT20] and [PTklef21], with Palezzato, we studied the weak and strong k -Lefschetz properties, and the related notion of almost revlex ideal, in the non-Artinian case, proving that several results known in the Artinian case hold also in this more general setting. We also connected for the first time the study of Lefschetz properties to the study of sectional matrices, using the work we did in [BPT18]. A graded ring is said to have the weak Lefschetz property if the multiplication by a general linear form has maximal rank in any degree, and similarly, to have the strong Lefschetz property if the multiplications by every power of a general linear form have maximal rank in any degree. These properties have been intensively studied as can be seen by the vast bibliography, and they are connected to many topics in algebraic geometry, commutative algebra and combinatorics. Some of these connections are quite surprising and there are still several open questions. We refer to [HMMNWW13] for an overview of the Lefschetz properties in the Artinian case, and to [MN13] for several open questions in the area. We then applied these results to the study of the Jacobian algebra of hyperplane arrangements, with particular attention to the class of free arrangements, and we proved that if an arrangement is free then its Jacobian algebra has the strong Lefschetz property.

In [Tor22], I extended the work of [PTfree20] and I investigated the relations between freeness over a field of finite characteristic and over the rational for the class of multiarrangements of hyperplanes.

From the start of the grant, I also expanded my work on domination problems for graphs. We refer to [HHP98a], [HHP98b] and [HY13] for a comprehensive account of this subject.

With Pahlavsay and Palezzato, in [PPT21], we computed lower and upper bounds for the domination number and the k -tuple total domination numbers of latin square graphs and described a formula for the 2-tuple total domination number.

With Kazemnejad, Pahlavsay and Palezzato, in [KPPT22], [KPPTtot23] and [KPPTdomin23], we described the domination, the total domination and the total dominator coloring numbers of the middle graph associated to a simple graph. In particular, we obtained several tight bounds for the domination, the total domination and the total dominator coloring numbers in terms of the order of the graph. We also computed the the domination, the total domination and the total dominator coloring numbers of the middle graph of several known families of graphs explicitly. Moreover, we presented some Nordhaus-Gaddum-like relations for the domination, the total domination and the total dominator coloring numbers of middle graphs.

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5. 主な発表論文等

〔雑誌論文〕 計15件（うち査読付論文 15件 / うち国際共著 14件 / うちオープンアクセス 1件）

1. 著者名 Torielli Michele	4. 巻 50
2. 論文標題 Freeness for multiarrangements of hyperplanes over arbitrary fields	5. 発行年 2021年
3. 雑誌名 Communications in Algebra	6. 最初と最後の頁 292 ~ 299
掲載論文のDOI (デジタルオブジェクト識別子) 10.1080/00927872.2021.1956514	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 -

1. 著者名 Palezzato Elisa, Torielli Michele	4. 巻 590
2. 論文標題 k-Lefschetz properties, sectional matrices and hyperplane arrangements	5. 発行年 2022年
3. 雑誌名 Journal of Algebra	6. 最初と最後の頁 215 ~ 233
掲載論文のDOI (デジタルオブジェクト識別子) 10.1016/j.jalgebra.2021.10.014	査読の有無 有
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1. 著者名 Kazemnejad Farshad, Pahlavsay Behnaz, Palezzato Elisa, Torielli Michele	4. 巻 0
2. 論文標題 Total dominator coloring number of middle graphs	5. 発行年 2022年
3. 雑誌名 Discrete Mathematics, Algorithms and Applications	6. 最初と最後の頁 1-12
掲載論文のDOI (デジタルオブジェクト識別子) 10.1142/S1793830922500768	査読の有無 有
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1. 著者名 Kazemnejad Farshad, Pahlavsay Behnaz, Palezzato Elisa, Torielli Michele	4. 巻 0
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1. 著者名 Guo Weili, Torielli Michele	4. 巻 77(3)
2. 論文標題 On the Falk invariant of hyperplane arrangements attached to gain graphs	5. 発行年 2020年
3. 雑誌名 Australasian Journal of Combinatorics	6. 最初と最後の頁 301~317
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1. 著者名 Torielli Michele, Tsujie Shuhei	4. 巻 27
2. 論文標題 Freeness of Hyperplane Arrangements between Boolean Arrangements and Weyl Arrangements of Type B _l	5. 発行年 2020年
3. 雑誌名 The Electronic Journal of Combinatorics	6. 最初と最後の頁 1-2
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1. 著者名 Palezzato Elisa, Torielli Michele	4. 巻 49
2. 論文標題 Localization of plus-one generated arrangements	5. 発行年 2020年
3. 雑誌名 Communications in Algebra	6. 最初と最後の頁 301 ~ 309
掲載論文のDOI (デジタルオブジェクト識別子) 10.1080/00927872.2020.1798976	査読の有無 有
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1. 著者名 Guo Weili, Torielli Michele	4. 巻 0
2. 論文標題 On the Falk Invariant of Shi and Linial Arrangements	5. 発行年 2021年
3. 雑誌名 Discrete & Computational Geometry	6. 最初と最後の頁 1-2
掲載論文のDOI (デジタルオブジェクト識別子) 10.1007/s00454-020-00266-0	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1. 著者名 Palezzato Elisa、Torielli Michele	4. 巻 128
2. 論文標題 Combinatorially equivalent hyperplane arrangements	5. 発行年 2021年
3. 雑誌名 Advances in Applied Mathematics	6. 最初と最後の頁 1-2
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1. 著者名 Pahlavsay Benhaz, Palezzato Elisa、Torielli Michele	4. 巻 0
2. 論文標題 Domination in latin square graphs	5. 発行年 2021年
3. 雑誌名 Graphs and Combinatorics	6. 最初と最後の頁 1-2
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1. 著者名 Palezzato Elisa、Torielli Michele	4. 巻 9
2. 論文標題 Hyperplane arrangements in CoCoA	5. 発行年 2019年
3. 雑誌名 Journal of Software for Algebra and Geometry	6. 最初と最後の頁 43 ~ 54
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1. 著者名 Palezzato Elisa、Torielli Michele	4. 巻 0
2. 論文標題 Free hyperplane arrangements over arbitrary fields	5. 発行年 2019年
3. 雑誌名 Journal of Algebraic Combinatorics	6. 最初と最後の頁 1-2
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1. 著者名 Pahlavsay Behnaz, Palezzato Elisa, Torielli Michele	4. 巻 0
2. 論文標題 3-tuple total domination number of rook's graphs	5. 発行年 2019年
3. 雑誌名 Discussiones Mathematicae Graph Theory	6. 最初と最後の頁 1-2
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1. 著者名 Settepanella Simona, Torielli Michele	4. 巻 75
2. 論文標題 On an explicit correspondence between nbc basis, chambers and minimal complex for real supersolvable arrangements	5. 発行年 2019年
3. 雑誌名 Australasian Journal of Combinatorics	6. 最初と最後の頁 223 ~ 245
掲載論文のDOI (デジタルオブジェクト識別子) なし	査読の有無 有
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1. 著者名 Palezzato Elisa, Torielli Michele	4. 巻 555
2. 論文標題 Lefschetz properties and hyperplane arrangements	5. 発行年 2020年
3. 雑誌名 Journal of Algebra	6. 最初と最後の頁 289 ~ 304
掲載論文のDOI (デジタルオブジェクト識別子) 10.1016/j.jalgebra.2020.02.039	査読の有無 有
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〔学会発表〕 計5件 (うち招待講演 3件 / うち国際学会 0件)

1. 発表者名 Michele Torielli
2. 発表標題 Hyperplane arrangements and k-Lefschetz properties
3. 学会等名 Special session "Algebraic Geometry from an Algorithmic Point of View" at the 26th Conference on Applications of Computer Algebra (ACA 2021)
4. 発表年 2021年

1. 発表者名 Michele Torielli
2. 発表標題 Comprehensive Groebner systems in CoCoA
3. 学会等名 Special session “Algorithms for Polynomial System Solving and Their Applications” at the 26th Conference on Applications of Computer Algebra (ACA 2021)
4. 発表年 2021年

1. 発表者名 Torielli Michele
2. 発表標題 Combinatorially equivalent hyperplane arrangements
3. 学会等名 Workshop “Recent advances in matroids and Tutte polynomials” (招待講演)
4. 発表年 2019年

1. 発表者名 Torielli Michele
2. 発表標題 Associated primes and localization of hyperplane arrangements
3. 学会等名 Workshop “Hyperplane arrangements in Wakkanai (招待講演)
4. 発表年 2019年

1. 発表者名 Torielli Michele
2. 発表標題 A modular approach to hyperplane arrangements
3. 学会等名 Workshop “Hyperplane Arrangements and Reflection Groups” (招待講演)
4. 発表年 2019年

〔図書〕 計0件

〔産業財産権〕

〔その他〕

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6. 研究組織

	氏名 (ローマ字氏名) (研究者番号)	所属研究機関・部局・職 (機関番号)	備考
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7. 科研費を使用して開催した国際研究集会

〔国際研究集会〕 計0件

8. 本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関
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