様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

科学研究費助成事業　　研究成果報告書

令和　６ 年　６ 月 １１ 日現在

機関番号：１３９０１
研究種目：若手研究
研究期間：2019～2023
課題番号：１９Ｋ１４５４８

研究課題名（和文）Interacting topological phases and operator algebras

研究課題名（英文）Interacting topological phases and operator algebras

研究代表者
ボーン　クリストファー・ジャック（Bourne, Christopher Jack）

名古屋大学・教養教育院・准教授

研究者番号：２０８３０１１０

交付決定額（研究期間全体）：（直接経費）　　2,800,000円

研究成果の概要（和文）：本プロジェクトはギャップのある基底状態のトポロジカル的な性質を研究することです．すなわち，低エネルギーの量子力学において少し変化しても，不変量の性質です．作用素環，非可換指数理論といった枠組みを用いて研究しました．
作用素環のホモロジー・コホモロジーを使用して，さまざまなギャップのある基底状態の位相的な性質を明らかにしました．

研究成果の学術的意義や社会的意義
Ground states give the most basic information about quantum mechanical system. By understanding the topological properties of ground states, we can understand which systems can be loosely connected and which are manifestly distinct. This aids our conceptual understanding of materials.

研究成果の概要（英文）：The project was an exploration of topological properties of gapped ground states. That is, properties of low-energy quantum mechanical systems which are stable under small perturbations and deformations. Our primary method for studying such problems was to use methods from operator algebras and non-commutative index theory.
Homology and cohomology are mathematical tools that give a simple algebraic description of a potential complicated setting (for example, how many holes in a shape). By using homology and cohomology theories for operator algebras, which describe quantum mechanical systems, we mathematically characterised stable properties of a wide variety of gapped ground states.

研究分野：作用素環論

キーワード：Topological phase　Operator algebras　Index theory　作用素環論　トポロジカル相　指数理論

1 Background leading to to research project

At the beginning of the research project (2019), topological phases of matter was an active area of research in physics and was gaining further attention in the mathematical physics community. The field's mathematical developments could loosely be separated into two related but distinct categories:

(i) *Topological* properties of free-fermionic and symmetric Hamiltonians with a spectral gap (topological insulators and superconductors),

(ii) *Analytic* properties of the many-body ground states of lattice systems with a spectral gap above the ground state energy in the infinite volume limit.

Both areas of research are concerned with showing certain invariance properties of mathematical objects that can be constructed in their respective setting. Examples include the stability of currents localised at the boundary of a free-fermionic Hamiltonian or the stability of the spectral gap of many-body ground states under perturbations of the many-body interaction.

While (i) and (ii) have important implications in condensed matter physics with potential applications in topological quantum computing, their mathematical techniques are quite distinct. At the time, it was unclear to what extent ideas from one area of study may be of relevance to the other.

2 Purpose of research

The central aim of the research project was to investigate the links between the research areas (i) and (ii) by investigating topological properties of a special class of many-body systems. In more detail, the broad aims of the project were the following:

(a) Expand our understanding of topological phases of matter to systems with many-body interactions,

(b) Investigate and establish new connections between the topological and analytical approaches to gapped systems,

(c) Prove theorems of independent mathematical interest.

The Principal Investigator (PI) had previously studied on topological phases of free-fermionic Hamiltonians. The project aimed to expand this knowledge to a much wider class of important examples.

3 Research methods

To make precise both the definition of gapped many-body ground state as well as its phase, it is desirable to work in the infinite volume limit. Such a setting necessitates the use of operator algebraic methods. In certain simpler settings, gapped ground states can also be characterised via certain elements of a C^*-algebra, for example the Fermi projection of a free-fermionic Hamiltonian. Hence, the project was primarily focused in studying properties of a special class of states or elements of a C^*-algebra that characterises the physical system.

In order to extract topological properties of states or elements of a C^*-algebra, one must study topological properties of the underlying algebra. Starting with a C^*-algebra A, there are a variety of mathematical tools to study its topological properties. One such example are the K-theory and K-homology groups of A, which then also provides potential connections to index theory and noncommutative geometry. In another setting, one may be interested in characterising symmetric Hamiltonians or ground states, which leads to the study of group actions on operator algebras. Topologically characterising such group actions may also lead to connections with group cohomology or other cohomology theories.

Depending on the system of study, a variety of approaches can be used to both define and characterise its topological phase. Though all of these approaches have a natural description in the language of operator algebras. As such, the projects goals were investigated using the framework of operator algebras, index theory and cohomology.

4 Research results

Let us give a brief summary of the results obtained through the KAKENHI project.

4.1 Results on gapped ground states

As a first step to consider more general gapped ground states, one-dimensional fermionic systems were considered. That is, a class of ground states on the infinite one-dimensional canonical anti-commutation relations (CAR) algebra $A^{\mathrm{car}}_{\mathbb{Z}}$. An important difference between the fermionic and bosonic systems, which are described via quantum spin chains, is that the CAR algebra comes with a canonical $\mathbb{Z}_2$-grading from the fermionic parity. It is physically reasonable to restrict one's attention to gapped ground states that respect this fermionic parity. Building from work by Matsui [10], a $\mathbb{Z}_2$-valued index was defined for such parity-symmetric gapped ground states on $A^{\mathrm{car}}_{\mathbb{Z}}$ and its basic stability properties were shown in the paper [7]. This work was further developed and generalised in [6], which defined a group cohomology-valued index for gapped fermionic ground states in one dimension and with an additional finite group symmetry G. Specifically, the index takes value in $\mathbb{Z}_2 \times H^1(G, \mathbb{Z}_2) \times H^2(G, U(1))$, where $H^\bullet(G, K)$ denote the group cohomology with coefficients in K. This index is indeed an invariant of gapped ground states and so can be considered as the topological phase of G-symmetric fermionic gapped ground states in one dimension.

We also remark that the paper [6] also develops the theory of fermionic matrix product states, which provide a computationally convenient approach to study translation-symmetric gapped ground states in one dimension.

More general gapped ground states of fermionic systems, e.g. in higher dimension, are studied via states on the more general CAR algebra $A^{\mathrm{car}}(\mathcal{H})$, where $\mathcal{H}$ describes the underlying system. Such a class of states is extremely large, so it is convenient to either restrict the underlying setting or the type of ground state. A particularly amenable class of ground states are the so-called quasifree states of $A^{\mathrm{car}}(\mathcal{H})$, which have a close relationship to the Bogoliubov–de Gennes description of superconductors, an intrinsically many-body phenomenon. In the paper [1], we studied G-symmetric quasifree ground states with reference to an underlying space M. Using the space M, a notion of locally equivalent states was considered and it was shown that there is a topological obstruction for locally equivalent states homotopically connected. This obstruction takes values in the G-symmetric K-homology of the space M, $K^G_*(M)$. Of particular note is that both M and G can be non-compact. The coarse index, an index-theoretic map for non-compact spaces, can also be used to relate this K-homology index to the K-theoretic indices that are constructed for superconducting systems via the K-theory Roe C^*-algebras.

4.2 Mathematical results

The project's analysis of topological phases and gapped ground states also gave rise to more mathematical questions and studies. While motivated and with applications in the project's physical systems, these results are of independent mathematical interest.

The study of fermions and the CAR algebra requires a careful study of operator algebras with a $\mathbb{Z}_2$-grading. An approach to K-theory for $\mathbb{Z}_2$-graded operator algebras that is particularly amenable to the study of gapped Hamiltonians was introduced by van Daele [8]. An explicit connection between van Daele K-theory and more standard approaches to operator algebraic K-theory via Kasparov's KK-theory had not been shown. The paper [4] fills this gap via an explicit isomorphism using a $\mathbb{Z}_2$-graded version of the Cayley transform of a self-adjoint or unitary operator. The relevance of this map to bulk and edge systems of free-fermionic topological phases was also shown.

The study of topological obstructions to homotopies of quasifree gapped ground states of the CAR algebra in [1] motivated a more general study of spectral flow of the (skew-adjoint) Fredholm operators in real Hilbert spaces with Clifford symmetries. A comprehensive analysis of this question was carried out in the paper [3], where it was also shown that the spectral flow can be described as the via the skew-adjoint Fredholm index of a Dirac-like Hamiltonian.

Quasifree ground states of the CAR algebra with a $U(1)$-symmetry can be characterised by the spectral projection of a Hamiltonian. If the Hamiltonian is a Schrödinger operator that with a potential that is periodic with respect to an underlying lattice, the Bloch–Floquet transform can be used to give a set of functions whose lattice translations generate an orthonormal basis of this spectral subspace. Interestingly, the regularity of these functions can be used to determine if the spectral subspace and corresponding ground state is topologically trivial or not. In the paper [5], this result is extended to aperiodic potentials, where the Bloch–Floquet transform is unavailable. This result is shown by constructing a discrete groupoid $\mathcal{G}$ and then studying the K-theory of the corresponding groupoid C^*-algebra.

4.3 Other results

The goal of the project is to construct and understand topological indices for a gapped Hamiltonians and ground states in a variety of settings. Methods from quantum information theory have also been quite effective in understanding further properties of ground states. As such, we also began to investigate topological properties of a few simple descriptions of time evolutions and operations on quantum states. Quantum walks provide a flexible model for the discrete time step of a Hamiltonian. In the paper [2], we defined topological indices for quantum walks with an additional chiral symmetry in the very general setting of Hilbert C^*-modules. This result also has implications for edge/boundary properties of Hamiltonians on half-space systems.

These results in [2] can be considered as the first steps towards a larger study of the relevance of methods from index theory and noncommutative geometry in the study of quantum information theory.

Discussion and conclusion

With regards to the projects desired outcomes and obtained results, we are generally quite satisfied with what we have been able to accomplish. Namely, we were able to characterise the topological phase of gapped ground states in a wide variety of different settings. A limitation of our results is that we were only able to obtain results for general many-body ground states in one dimension. Understanding topological phases of higher dimensional many-body systems seems to require further restrictions on the types of gapped ground states considered (namely, ground states that are homotopic to a product state) as well as quite different mathematical techniques, cf. [9]. Another possible direction for new research is to consider higher-dimensional gapped ground states, but where the allowed deformations are much more restricted and controlled by some auxiliary C^*-algebra contained within the CAR algebra.

The new connections between index theory and quantum walks considered in [2] also open up new directions to study the connections between index theory and systems in quantum information theory. We aim to further study this question in future work, which may also be relevant for developing our understanding of operations and elementary excitations of gapped ground states. Such excitations have been proposed as a mechanism for constructing quantum computational algorithms. Clarifying some of these questions may therefore be of much broader significance.

References

[1] Bourne, C.: Locally equivalent quasifree states and index theory. J. Phys. A: Math. Theor., **55**, no. 10, Paper No. 104004, 38 pp. (2022).

[2] Bourne, C.: Index theory of chiral unitaries and split-step quantum walks. SIGMA, **19**, Article No. 053, 39 pages (2023).

[3] Bourne, C., Carey, A.L., Lesch, M., Rennie, A.: The KO-valued spectral flow for skew-adjoint Fredholm operators. J. Topol. Anal., **14**, no. 2, 505–556 (2022).

[4] Bourne, C., Kellendonk, J., Rennie, A.: The Cayley transform in complex, real and graded K-theory. Internat. J. Math., **31**, no. 9, 2050074, 50 pages (2020).

[5] Bourne, C., Mesland, B.: Localised module frames and Wannier bases from groupoid Morita equivalences. J. Fourier Anal. Appl., **27**, no. 4, Paper No. 69, 39 pages (2021).

[6] Bourne, C., Ogata, Y.: The classification of symmetry protected topological phases of one-dimensional fermion systems. Forum Math. Sigma, **9**, Paper No. e25, 45 pages (2021).

[7] Bourne, C., Schulz-Baldes, H.: On $\mathbb{Z}_2$-indices for ground states of fermionic chains. Rev. Math. Phys., **32**, no. 9, 2050028, 69 pages (2020).

[8] van Daele, A.: K-theory for graded Banach algebras I. Quart. J. Math. Oxford Ser. (2), **39**, no. 154, 185–199 (1988).

[9] Ogata, Y.: An $H^3(G, \mathbb{T})$-valued index of symmetry-protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems. Forum Math. Pi, **9**, Paper No. e13, 62 pages (2021).

[10] Matsui, T.: Boundedness of entanglement entropy and split property of quantum spin chains. Rev. Math. Phys., **25**,, no. 9, 1350017, 31 pages (2013).

５．主な発表論文等

〔雑誌論文〕　計7件（うち査読付論文　7件／うち国際共著　4件／うちオープンアクセス　2件）

１．著者名 Bourne Chris、Mesland Bram	４．巻 27
２．論文標題 Localised Module Frames and Wannier Bases from Groupoid Morita Equivalences	５．発行年 2021年
３．雑誌名 Journal of Fourier Analysis and Applications	６．最初と最後の頁 1-39
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00041-021-09873-8	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Bourne Chris	４．巻 55 (104004)
２．論文標題 Locally equivalent quasifree states and index theory	５．発行年 2022年
３．雑誌名 Journal of Physics A: Mathematical and Theoretical	６．最初と最後の頁 1-38
掲載論文のDOI（デジタルオブジェクト識別子） 10.1088/1751-8121/ac508b	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Bourne Chris、Kellendonk Johannes、Rennie Adam	４．巻 31
２．論文標題 The Cayley transform in complex, real and graded K-theory	５．発行年 2020年
３．雑誌名 International Journal of Mathematics	６．最初と最後の頁 1~50
掲載論文のDOI（デジタルオブジェクト識別子） 10.1142/S0129167X20500743	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Bourne Chris、Ogata Yoshiko	４．巻 9
２．論文標題 The classification of symmetry protected topological phases of one-dimensional fermion systems	５．発行年 2021年
３．雑誌名 Forum of Mathematics, Sigma	６．最初と最後の頁 1~45
掲載論文のDOI（デジタルオブジェクト識別子） 10.1017/fms.2021.19	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

1．著者名 Bourne Chris、Schulz-Baldes Hermann	4．巻 32
2．論文標題 On Z_2-indices for ground states of fermionic chains	5．発行年 2020年
3．雑誌名 Reviews in Mathematical Physics	6．最初と最後の頁 2050028-1～69
掲載論文のDOI（デジタルオブジェクト識別子） 10.1142/S0129055X20500282	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Bourne Chris、Carey Alan L.、Lesch Matthias、Rennie Adam	4．巻 14
2．論文標題 The KO-valued spectral flow for skew-adjoint Fredholm operators	5．発行年 2020年
3．雑誌名 Journal of Topology and Analysis	6．最初と最後の頁 505～556
掲載論文のDOI（デジタルオブジェクト識別子） 10.1142/S1793525320500557	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Bourne Chris	4．巻 19 (53)
2．論文標題 Index Theory of Chiral Unitaries and Split-Step Quantum Walks	5．発行年 2023年
3．雑誌名 Symmetry, Integrability and Geometry: Methods and Applications	6．最初と最後の頁 1-39
掲載論文のDOI（デジタルオブジェクト識別子） 10.3842/SIGMA.2023.053	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

〔学会発表〕　計30件（うち招待講演　27件／うち国際学会　15件）

1．発表者名 Chris Bourne
2．発表標題 Split-step quantum walks, chiral unitaries and K-theory
3．学会等名 University of Wollongong Operator Algebras and Noncommutative Geometry Seminar（招待講演）（国際学会）
4．発表年 2022年

１．発表者名
Chris Bourne

２．発表標題
Fermionic gapped ground states, topological phases and index theory

３．学会等名
Winter School and Conference on Noncommutative Geometry and Geometric Analysis（招待講演）（国際学会）

４．発表年
2022年

１．発表者名
Chris Bourne

２．発表標題
Index theory and topological phases for symmetric quantum walks

３．学会等名
CREST Research Seminar on "Theoretical studies of topological phases of matter"（招待講演）

４．発表年
2023年

１．発表者名
Chris Bourne

２．発表標題
SPT phases of quaisfree ground states and coarse geometry

３．学会等名
Oxford University QFT Seminar Series（招待講演）（国際学会）

４．発表年
2022年

１．発表者名
Chris Bourne

２．発表標題
Fermionic gapped ground states and topological phases in the infinite volume limit

３．学会等名
CREST Research Seminar on "Theoretical studies of topological phases of matter"（招待講演）

４．発表年
2021年

1．発表者名 Chris Bourne
2．発表標題 Locally equivalent quasifree states and the coarse index
3．学会等名 65th Annual Meeting of the Australian Mathematics Society（国際学会）
4．発表年 2021年

1．発表者名 Chris Bourne
2．発表標題 Symmetry protected topological (SPT) phases of infinite fermion chains
3．学会等名 65th Annual Meeting of the Australian Mathematics Society（国際学会）
4．発表年 2021年

1．発表者名 Chris Bourne
2．発表標題 Localized/de-localized Wannier bases of aperiodic Schroedinger operators
3．学会等名 ICMP Satellite Workshop: Topological Phases of Matter（招待講演）（国際学会）
4．発表年 2021年

1．発表者名 Chris Bourne
2．発表標題 KK-theory and free fermionic topological phases
3．学会等名 Simon's Center online workshop: C*-algebras, K-theories and Noncommutative Geometries of Correlated Condensed Matter Systems（招待講演）（国際学会）
4．発表年 2021年

1．発表者名
Chris Bourne

2．発表標題
Aperiodic and amorphous topological phases

3．学会等名
RIKEN iTHEMS Quantum Matter Study Group Seminar（招待講演）

4．発表年
2021年

1．発表者名
Chris Bourne

2．発表標題
Introduction to the mathematics of (aperiodic) topological materials

3．学会等名
Joint Applied Math Online Seminar（招待講演）

4．発表年
2020年

1．発表者名
Chris Bourne

2．発表標題
Symmetry protected topological (SPT) phases of infinite fermion chains

3．学会等名
The Second Joint RIKEN-Tohoku Workshop（招待講演）（国際学会）

4．発表年
2020年

1．発表者名
Chris Bourne

2．発表標題
Symmetry protected topological (SPT) phases of infinite fermion chains

3．学会等名
Wales Mathematical Physics - Physical Mathematics Seminar（招待講演）（国際学会）

4．発表年
2020年

１．発表者名
Chris Bourne

２．発表標題
Wannier bases and topology of aperiodic Schroedinger operators

３．学会等名
Rigorous Statistical Mechanics and Related Topics II（招待講演）（国際学会）

４．発表年
2020年

１．発表者名
Chris Bourne

２．発表標題
Symmetry protected topological (SPT) phases of infinite fermion chains

３．学会等名
Wollongong Operator Algebra and Noncommutative Geometry Seminar（招待講演）（国際学会）

４．発表年
2021年

１．発表者名
Chris Bourne

２．発表標題
Z_2-indices of ground states of fermionic chains

３．学会等名
Topological Phases of Interacting Systems（招待講演）

４．発表年
2019年

１．発表者名
Chris Bourne

２．発表標題
Groupoid cocycles and Delone dynamical systems

３．学会等名
Groupoids, Graphs and Algebras

４．発表年
2019年

１．発表者名
Chris Bourne

２．発表標題
The Cayley transform and K-theory

３．学会等名
Operator Algebras and Noncommutative Geometry seminar, University of Wollongong（招待講演）

４．発表年
2019年

１．発表者名
Chris Bourne

２．発表標題
The Cayley transform and K-theory

３．学会等名
University of Erlangen-Nuremberg Mathematical Physics Seminar（招待講演）

４．発表年
2020年

１．発表者名
Chris Bourne

２．発表標題
The Cayley transform in complex, real and graded K-theory

３．学会等名
Bonn University Seminar in Global Analysis and Operator Algebras（招待講演）

４．発表年
2020年

１．発表者名
Chris Bourne

２．発表標題
Spectral flow, index theory and free-fermionic topological phases

３．学会等名
Cologne University Theoretical Physics Seminar（招待講演）

４．発表年
2020年

1．発表者名 Chris Bourne
2．発表標題 Spectral flow and topological phases
3．学会等名 Noncommutative Geometry, Analysis, and Topological Insulators（招待講演）
4．発表年 2020年

1．発表者名 Chris Bourne
2．発表標題 Spectral flow and topological phases
3．学会等名 Topological phenomena in non-Hermitian and non-equilibrium systems（招待講演）
4．発表年 2020年

1．発表者名 Chris Bourne
2．発表標題 Topological indices/phases of gapped fermionic ground states
3．学会等名 Workshop on Operator Algebras, Deformation Quantization and Related Field Theories（招待講演）（国際学会）
4．発表年 2023年

1．発表者名 Chris Bourne
2．発表標題 Topological stability of quantum walks and related
3．学会等名 10th International Congress on Industrial and Applied Mathematics (ICIAM)（招待講演）（国際学会）
4．発表年 2023年

１．発表者名 Chris Bourne
２．発表標題 An operator algebraic approach to quantum walks
３．学会等名 Studies on long time behavior for quantum walks（招待講演）
４．発表年 2023年

１．発表者名 Chris Bourne
２．発表標題 量子ウォークへの作用素環的なアプローチ
３．学会等名 作用素論・作用素環論研究集会（招待講演）
４．発表年 2023年

１．発表者名 Chris Bourne
２．発表標題 Spectral properties of quantum walk unitaries
３．学会等名 ANU Partial Differential Equations and Analysis Seminar（招待講演）（国際学会）
４．発表年 2024年

１．発表者名 Chris Bourne
２．発表標題 Operator algebras and index theory in quantum walks
３．学会等名 Noncommutative Geometry, Analysis on Groups, and Mathematical Physics（招待講演）（国際学会）
４．発表年 2024年

１．発表者名 Chris Bourne
２．発表標題 Spectral and topological properties of quantum walks
３．学会等名 Himeji Conference on Partial Differential Equations（招待講演）
４．発表年 2024年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

Personal academic webpage https://sites.google.com/site/khomologyzone/home

６．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考
研究協力者	Ｓｃｈｕｌｚ－Ｂａｌｄｅｓ　Ｈｅｒｍａｎｎ (Schulz-Baldes Hermann)		
研究協力者	Ｍｅｓｌａｎｄ　Ｂｒａｍ (Mesland Bram)		
研究協力者	Ｒｅｎｎｉｅ　Ａｄａｍ (Rennie Adam)		

６．研究組織（つづき）

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考
研究協力者	Ｋｅｌｌｅｎｄｏｎｋ　Ｊｏｈａｎｎｅｓ (Kellendonk Johannes)		
研究協力者	緒方　芳子 (Ogata Yoshiko)		
研究協力者	Ｃａｒｅｙ　Ａｌａｎ (Carey Alan)		
研究協力者	Ｌｅｓｃｈ　Ｍａｔｔｈｉａｓ (Lesch Matthias)		

７．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

８．本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関