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研究課題名（和文）The projective geometry of Zoll surfaces and the Cut locus on Finsler manifolds

研究課題名（英文）The projective geometry of Zoll surfaces and the Cut locus on Finsler manifolds

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研究成果の概要（和文）：本研究は、Zoll計量によって誘導される一定の正の旗曲率を持つFinslerian計量の幾何学的性質の研究と距離関数とcut locusの性質を使用したFinslerian多様体の幾何学と位相幾何学の研究を含んでいる。Randers型の回転面上で、測地線の局所的および大域的な振る舞い、cut locusの構造を決定した。言語SAGEを使ってコンピューター上で数値シミュレーションを実行した。
様々な回転面上のRanders型計量のcut locusを研究し、測地線の振る舞いとcut locusの構造は、Killing風を伴うZermelo航行ケースよりもはるかに一般的な場合で明示的に決定できます。

研究成果の学術的意義や社会的意義
Our reseach is important from scientific point of view because it develops more general geometrical concepts than the Riemannian ones showing that the real world is Finslerian.
From social point of view, brings together researchers from Asia and from USA and Europe in international conferences.

研究成果の概要（英文）：The present research include The study the geometrical and topological properties of Finsler metrics of constant positive flag curvature induced by Zoll metrics, and The study the geometry and topology of Finsler manifolds by using the properties of distance function and the cut locus. Some results are published already,others still in print.
We have determined the local and global behaviour of geodesics, the difference with the Riemannian case and the structure of the cut locus on a Randers surface of revolution.We have performed numerical simulation on computer using the programming language SAGE.
We have studied the cut locus of Randers type metrics on different surfaces of revolution, we have determined the local and global behaviour of geodesics,the structure of the cut locus using a Hamiltonian formalism. The geodesics behaviour and the structure of the cut locus can be explicitly determined in a much more general case than the Zermelo's navigation case with Killing wind.

研究分野：Differential geometry

キーワード：Riemannian manifolds　Finsler manifolds　surfaces of revolution　Killing vector fields　the theory of geodesics　cut locus　Zoll metrics　manifold of geodesics

様式 C-19, F-19-1 （共通）

1. 研究開始当初の背景

A Randers metric $F = \alpha + \beta$ is a special Finsler metric obtained by the deformation of a Riemannian metric α by a one-form β whose Riemannian α-length is less than one in order to assure that F is positive defined.

An equivalent characterization is through the *Zermelo's navigation problem. Consider a ship sailing on the open sea in calm waters. If a mild breeze comes up, how should the ship be steered in order to reach a given destination in the shortest time possible?* We recall that a Finsler metric F is characterized by its indicatrix $\{(x, y) \in TM : F(x, y) = 1\}$ (see [BCS]). In particular, a Randers metric indicatrix is obtained as the rigid translation of the unit sphere $\{y \in T_xM : h(x, y) = 1\}$ of a Riemannian metric (M, h) by a vector field $W \in \mathcal{X}(M)$ whose length is less than one. The pair (h, W) will be called the *navigation data* of the Randers metric $F = \alpha + \beta$. Conversely, the Randers metric $F = \alpha + \beta$ will be called the *solution of Zermelo's navigation problem* (h, W). In the case of Killing field wind, the geodesics, cut points, and conjugate points of the Randers metric $F = \alpha + \beta$ can be obtained by the translation of the corresponding geodesics, cut points, and conjugate points of the Riemannian metric h by the flow of W, respectively (see [HS1], [R]). More generally, new Finsler metrics F can be obtained by the rigid translation of the indicatrix of a given Finsler metric F_0 by a vector field W. In this case, the pair (F_0, W) will be called the *general navigation data* of F.

2. 研究の目的

The purpose of this research project is two folded.

(I) Study of the conjugate and cut locus of Randers metrics $F = \alpha + \beta$ defined by the Zermelo's navigation in more general cases than the one where the wind is Killing vector field.

(II) Study the geometry of a Finsler manifold of scalar curvature which are not of constant flag curvature by extending the results obtained in the case of Zoll metrics. We have started the research assuming that the Finsler metric is of Randers type, and

3. 研究の方法

We have used classical methods of Differential Geometry in order to perform this research. The geometry of Randers spaces is characterized by many local and global computations concerning geodesics, flag curvature, scalar curvature and many other geometrical objects. We have used hand computation for the most of these quantities and then we have double checked the computations using symbolic computation software like the package SAGE written in Python and others.

4. 研究成果　The results presented here are mainly from the papers [HS1], [HS2], [BHS].

(I) We will construct Randers metrics using the following basic construction.

$$(M,h)\xrightarrow[V_0:\ h\text{-Killing}]{V_0,\ \|V\|_h<1}(M,F_0=\alpha_0+\beta_0)\xrightarrow[V:\ F_0\text{-Killing}]{V,\ F_0(-V)<1}(M,F_1=\alpha_1+\beta_1)\xrightarrow[d\beta_2=0]{W,\ F_1(-W)<1}(M,F_2=\alpha_2+\beta_2)$$

Navigation data: (h,V_0) (h,V_0+V) (h,V_0+V+W)

where (M,h) is a Riemannian manifold, and $V_0, V, W \in \mathcal{X}(M)$ are vector fields on M.

Theorem 0.1 *Let (M,h) be a Riemannian manifold and let $V_0, V, W \in \mathcal{X}(M)$ be vector fields on M.*

(i)(i.1) If $\|V_0\|_h < 1$, then $F_0 = \alpha_0+\beta_0$ is a positive defined Randers metric, where F_0 is the solution of Zermelo's navigation problem with data (h, V_0).

(i.2) If $F_0(-V) < 1$, then $F_1 = \alpha_1 + \beta_1$ is a positive defined Randers metric, where F_1 is the solution of Zermelo's navigation problem with data (F_0, V).

(i.3) If $F_1(-W) < 1$, then $F_2 = \alpha_2 + \beta_2$ is a positive defined Randers metric, where F_2 is the solution of Zermelo's navigation problem with data (F_1, W).

(ii)(ii.1) The Randers metric $F_1 = \alpha_1 + \beta_1$ is the solution of Zermelo's navigation problem with data $(h, V_0 + V)$.

(ii.2) The Randers metric $F_2 = \alpha_2 + \beta_2$ is the solution of Zermelo's navigation problem with data $(h, V_0 + V + W)$.

(iii) If the following conditions are satisfied

(C0) V_0 is h-Killing

(C1) V is F-Killing

(C2) $d(V_0 + V + W)^\# = d\log\widetilde{\lambda} \wedge (V_0 + V + W)$, where $(V_0 + V + W)^\# = \mathcal{L}_h(V_0+V+W)$ is the Legendre transformation of V_0+V+W with respect to h, and $\widetilde{\lambda} := 1 - \|V_0 + V + W\|_h^2$, then

(iii.1) The F_0-unit speed geodesics $\mathcal{P}_0$, and the F_1-unit speed geodesics $\mathcal{P}_1$ are given by

$$\mathcal{P}_0(t) = \varphi_t(\rho(t)), \quad \mathcal{P}_1(t) = \psi_t(\mathcal{P}_0(t)) = \psi_t \circ \varphi_t(\rho(t)), \tag{0.1}$$

where $\rho(t)$ is an h-unit speed geodesic and φ_t and ψ_t are the flows of V_0, and V, respectively. The F_2-unit speed geodesic $\mathcal{P}_2(t)$ coincides as points set with $\mathcal{P}_1(t)$.

(iii.2) The conjugate points of $q = \mathcal{P}_2(0)$ along the F_2-geodesic $\mathcal{P}_2$ coincide to the conjugate points of $q = \mathcal{P}_1(0)$ along the F_1-geodesic $\mathcal{P}_1$, up to parametrization.

The point $\mathcal{P}_1(l)$ is conjugate to $q = \mathcal{P}_1(0)$ along the F_1-geodesic $\mathcal{P}_1 : [0,l] \to M$ if and only if the point $\mathcal{P}_0(l)$ is conjugate to $q = \mathcal{P}_0(0)$ along the corresponding F_0-geodesic $\mathcal{P}_0(t) = \psi_{-t}(\mathcal{P}_1(t))$, for $t \in [0,l]$.

The point $\mathcal{P}_0(l)$ is conjugate to $q = \mathcal{P}_0(0)$ along the F_0-geodesic $\mathcal{P}_0 : [0, l] \to M$ if and only if the point $\rho(l)$ is conjugate to $q = \rho(0)$ along the corresponding h-geodesic $\rho(t) = \varphi_{-t}(\mathcal{P}_0(t))$, for $t \in [0, l]$, where φ_t, and ψ_t are the flows of V_0, and V, respectively.

(iii.3) The F_2-cut locus of q coincide as points set with the F_1cut locus of q, up to parametrization.
The point $\hat{p}_1$ is an F_1-cut point of q, if and only if $\hat{p}_0 = \psi_{-l}(\hat{p}_1)$ is an F_1-cut point of q, where $l = d_{F_1}(q, \hat{p}_1)$. The point $\hat{p}_0$ is an F_0-cut point of q, if and only if $p_0 = \varphi_{-l}(\hat{p}_0)$ is an h-cut point of q, where $l = d_{F_0}(q, \hat{p}_0)$.

(II) The geometry of Randers metric of constant flag curvature is well understood. An important example of Finsler metric of constant curvature induced by a Zoll metric is presented in [KSS]. Some of the results described below have not been published yet.

In the case the flag curvature K do not depend on the transversal wedge V, the Finsler metric F is said to be of *scalar flag curvature.* Obviously, every 2-dimensional Finsler manifold is of scalar flag curvature, hence this definition makes sense for dimension ≥ 3 only.

Moreover, if K is a constant, then the Finsler metric F is called of *constant flag curvature.*

Theorem 0.2 *Let $(M = \mathbb{S}^n, \alpha = can_{\mathbb{S}^n})$ be the n-sphere with the canonical metric. Let $f : \mathbb{S}^n \to \mathbb{R}$ be a smooth function such that*

(i) $|\nabla f|_\alpha \neq constant$, $|\nabla f|_\alpha < 1$, where ∇f is the gradient vector field of f,

(ii) otherwise, ∇f is not Killing vector field on M.

Then the Randers metric $F = \alpha + df$ is of scalar flag curvature, but not of constant flag curvature.

Example 0.3 Let us start with $\mathbb{S}^3$ regarded as the doubly warped product $\left(0, \frac{\pi}{2}\right) \times \mathbb{S}^1 \times \mathbb{S}^1$ with coordinates (t, u, v) and the canonical Riemannian metric

$$\alpha^2 = dt^2 + \sin^2 t(du)^2 + \cos^2 t(dv)^2. \tag{0.2}$$

If we denote $\max := 4\pi^2 + \frac{\pi^2}{4}$, then

$$\beta := \frac{1}{\max}\widetilde{\beta} = \frac{1}{\max}df = \frac{1}{\max}(udt + tdu) \tag{0.3}$$

is a closed 1-form on $\mathbb{S}^3$, s.t. $|b|^2_\alpha < 1$.

Theorem 0.4 *Let us consider the Randers space $(M = \mathbb{S}^3, F = \alpha + \beta)$ where α and β are given by (0.2) and (0.3), respectively. Then F is of scalar flag curvature, but not constant flag curvature.*

We have also computed using symbolic computation in SAGE the scalar curvature of this metric and obtain a result that actually shows that this scalar is not constant.

Remark 0.5 In [BS] the authors have constructed right invariant Randers metric of constant flag curvature on the Lie-group SU(2) = $\mathbb{S}^3$. If we try to construct a right invariant Randers metric of scalar curvature, this is not possible. Indeed, let us consider the Lie Group $G = SU(2)$ with the Lie algebra $\mathfrak{su}(2)$.

First, we compute $d\beta(X, Y)$, where X, Y are right-invariant vector field on G:

$$\begin{aligned} d\beta(X, Y) &= X(\beta(Y)) - Y(\beta(X)) - \beta([X, Y]) \\ &= 0 - 0 - \beta([X, Y]) = -\beta([X, Y]), \end{aligned}$$

where we use that β is right invariant if and only if $\beta(X) =$ constant, for any right invariant vector field X.

Next, we assume $d\beta = 0$. This implies $\beta = 0$ because $[\mathfrak{su}(2), \mathfrak{su}(2)] = \mathfrak{su}(2)$. Therefore:

Theorem 0.6 *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. If β is a right invariant one-form satisfying $d\beta = 0$ everywhere, then $\beta \equiv 0$.*

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５．主な発表論文等

〔雑誌論文〕　計3件（うち査読付論文　3件／うち国際共著　3件／うちオープンアクセス　3件）

１．著者名 I. Masca, S. V. Sabau, H. Shimada	４．巻 T. LXVII, f. 1
２．論文標題 A Finsler metric of constant Gauss curvature K=1 on 2-sphere	５．発行年 2021年
３．雑誌名 Scientific Annals of the Alexandru Ioan Cuza University of Iasi (New Series), Mathematics	６．最初と最後の頁 31-44
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

１．著者名 Hama Rattanasak、Harko Tiberiu、Sabau Sorin V.、Shahidi Shahab	４．巻 81
２．論文標題 Cosmological evolution and dark energy in osculating Barthel?Randers geometry	５．発行年 2021年
３．雑誌名 The European Physical Journal C	６．最初と最後の頁 1-33
掲載論文のDOI（デジタルオブジェクト識別子） 10.1140/epjc/s10052-021-09517-7	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

１．著者名 Rattanasak Hama and Sorin V. Sabau	４．巻 8(11), 2047
２．論文標題 The Geometry of a Randers Rotational Surface with an Arbitrary Direction Wind	５．発行年 2020年
３．雑誌名 Mathematics	６．最初と最後の頁 1-27
掲載論文のDOI（デジタルオブジェクト識別子） 10.3390/math8112047	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

〔学会発表〕　計1件（うち招待講演　0件／うち国際学会　1件）

１．発表者名 Sabau V. Sorin
２．発表標題 On the variational problem for Kropina Manifolds
３．学会等名 The 55-th Symposium on Finsler geometry（国際学会）
４．発表年 2020年～2021年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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6．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

7．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

8．本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関