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研究成果の概要（和文）：私は基本群のモジュライ空間を定義し、位相同型予想を定式化することができた。また、1次元のモジュライ空間に対してこの位相同型予想が成立することを証明した。この基本群のモジュライ空間を使用して、正標数の代数閉体上の曲線に対する新たな遠アーベル哲学を提唱した。さらに、この哲学を基に、数々の新しい予想を定式化できた。そして、p進局所体の遠アーベル幾何においても応用が発見し、p進体上の曲線に関するGrothendieck予想のIsom版である望月氏の有名な定理に対して新たな証明に成功した。

研究成果の学術的意義や社会的意義
私の研究では、基本群のモジュライ空間という新たな理論を提唱し、いくつかの基本的な予想を定式化し、その予想が成り立つ証拠も提供した。基本群のモジュライ空間理論は、正標数の代数閉体上の曲線の遠アーベル幾何学に対する一般的な哲学を提供し、特にその主予想である位相同型予想は、今後の研究の方向性を指し示している。

研究成果の概要（英文）：I introduced a new topological space called the moduli space of fundamental groups and formulated the conjecture of topological isomorphism. I also proved that this topological isomorphism conjecture holds for one-dimensional moduli spaces. By using the moduli space of fundamental groups, I posed a new anabelian philosophy for curves over algebraically closed fields of positive characteristic. Furthermore, based on this philosophy, I formulated numerous new conjectures. As an application of the philosophy, I successfully provided a new proof for the famous theorem of Mochizuki concerning the Isom version of the Grothendieck conjecture for curves over sub-p-adic fields.

研究分野：代数幾何

キーワード：curve　moduli space　fundamental group　anabelian geometry　positive characteristic

1．研究開始当初の背景

I works in algebraic geometry and arithmetic geometry, for the most part, my research focuses on curves and their moduli spaces in positive characteristic from the point of view of fundamental groups. Recently, I discovered *a new kind of anabelian phenomenon* of curves in positive characteristic. This discovery demonstrates that fundamental groups of curves contain profoundly insightful information about moduli spaces of curves. Since anabelian geometry has been suggested by A. Grothendieck in 1980s, this is *the first anabelian phenomenon that cannot be explained by using Grothendieck's original anabelian philosophy.*

Building upon a multitude of results obtained by myself, I established a theory referred to as *"moduli spaces of fundamental groups".* This new theory provides a general framework for describing anabelian phenomena in positive characteristic, a topic extensively studied by F. Pop, M. Raynaud, M. Saïdi, and A. Tamagawa 27 years ago. Furthermore, this theory introduces *a new anabelian philosophy* that sheds light on understanding fundamental groups of curves in positive characteristic, contributing significantly to the comprehension of these fantastic mathematical objects.

Grothendieck's anabelian philosophy. In the 1980s, in his letter to G. Faltings and his famous manuscript "*La longue marche à travers la théorie de Galois*", Grothendieck suggested a new theory of arithmetic geometry called anabelian geometry. In the case of curves, Grothendieck's anabelian philosophy, roughly speaking, is as follows:

> *The sets of dominant morphisms of curves can be determined group-theoretically from the sets of open continuous homomorphisms of their algebraic fundamental groups.*

The conjectures based on this anabelian philosophy are called Grothendieck's anabelian conjectures, which have been deeply studied in the case of curves over "arithmetic fields" (e.g. number fields, p-adic fields, finite fields, etc.) by many mathematicians. Note that, in the case of arithmetic fields, *the non-trivial Galois actions* plays a central role in the theory of anabelian geometry.

Anabelian geometry of curves over algebraically closed fields of positive characteristic. Since the mid-1990s, some developments of F. Pop, M. Raynaud, M. Saïdi, A. Tamagawa, and me showed evidence for very strong *anabelian phenomena for curves over algebraically closed fields of characteristic* $p > 0$ (i.e. in positive characteristic world, anabelian geometry exists *without Galois actions*!). This kind of anabelian phenomenon is the reason why we do not have an explicit description of the geometric fundamental group of any curve in positive characteristic even the simplest case: $\pi_1(\mathbb{A}^1_{\overline{\mathbb{F}}_p})$. Moreover, we may think that the anabelian geometry of curves over k is a theory based on the following rough consideration: The admissible fundamental group of a curve over k must encode "*moduli*" of the curve.

2．研究の目的

In the theory of anabelian geometry of curves over algebraically closed fields in positive characteristic, one of the main conjectures is the so-called "*the weak Isom-version conjecture*" which was essentially formulated by Tamagawa 24 years ago in the case of smooth pointed stable curves, and which, recently, was extended by me in the case of arbitrary pointed stable curves. The weak Isom-version conjecture *shares the same anabelian philosophy as Grothendieck originally suggested.* Until I established the so-called "moduli spaces of fundamental groups" explained below, the weak Isom-version conjecture is the ultimate goal of anabelian geometry over algebraically closed fields of characteristic p and all of the researches focus on this conjecture.

3．研究の方法

In [15], I observed that the geometric behaviors of curves corresponding to the set of open continuous homomorphisms of fundamental groups are *not* the set of dominant morphisms but *the sets of deformations of curves* (as compared with the case of arithmetic fields). This observation implies *a new kind of anabelian phenomenon that cannot be explained by using Grothendieck's original anabelian philosophy*: The *topological structures of moduli spaces* of curves can be reconstructed group-theoretically from sets of fundamental groups of curves. This new anabelian phenomenon can be precisely captured by using the so-called *moduli spaces of fundamental groups* and *the homeomorphism conjecture* introduced by me.

4. 研究の成果

Establishing moduli spaces of fundamental groups. To establish the theory of moduli spaces of fundamental groups, since 2017, I introduced many new ideas and developed many new tools for the theory of anabelian geometry of curves in positive characteristic. For example, we have the following: In [5], [12], I developed a theory which we call "combinatorial anabelian geometry in positive characteristic". This theory allows us to treat anabelian geometry in positive characteristic not only for *smooth* curves but also for *arbitrary* (possibly singular) pointed stable curves. In [8], [11], [21], I studied the theory of Raynaud-Tamagawa theta divisor, in particular, we obtained two important formulas for the averages and the maximal generalized Hasse-Witt invariants which play central roles in the theory of moduli spaces of fundamental groups. In [9], I introduced the so-called "Frobenuis equivalence" on the moduli spaces of curves. It allows us to formulate the weak Isom-version conjecture for arbitrary pointed stable curves.

Based on the results obtained in [5], [8], [9], [11], [12], [15], in [17], I established the theory of moduli spaces of admissible fundamental groups and formulated the main conjecture: *the homeomorphism conjecture*. This new conjecture *generalizes all the conjectures* in the theory of anabelian geometry of curves over algebraically closed fields of characteristic p, and means that the moduli spaces of curves *can be reconstructed group-theoretically as topological spaces from sets of open continuous homomorphisms of fundamental groups* of curves.

At present, the homeomorphism conjecture has been proved by myself in the case where $\dim(\overline{M}_{g,n}) \leq 1$. This result is regarded as *the most significant progresses in the theory of anabelian geometry of curves over algebraically closed fields of characteristic* $p > 0$ *since 2001*. Particularly, it has revolutionized our perspective on anabelian geometry in positive characteristic and supplies a *new anabelian philosophy* towards Grothendieck's anabelian dream over algebraically closed fields of characteristic $p > 0$.

Moreover, leveraging this new anabelian philosophy, in [16], I (with Y. Hoshi) discovered an entirely new proof of Mochizuki's famous theorem (S. Mochizuki, *Invent. Math.* **138** (1999), 319-–423) concerning the Isom-version of Grothendieck's anabelian conjecture for curves over sub-p-adic fields which is one of the highest achievements of anabelian geometry.

Other results. In addition to the researches mentioned above, I also obtained many results concerning geometry of curves. For example, we have the following: In [1], I studied the ℓ-adic monodromy arsing a pointed stable curves and proved a certain analogue of the weight-monodromy conjecture for non-degenerate elements of pro-ℓ log étale fundamental groups of certain log points associated to log moduli stacks. In [2], [13], I studied the so-called "resolution non-singularities" phenomenon. I extended Tamagawa's result to the case of arbitrary local fields and obtained a formula for p-rank of non-finite fibers. The formula is a full generalization of Raynaud's formula. Moreover, by applying this formula, I solved an open problem concerning boundedness of p-rank which was posed by Saïdi. In [3], [10], I studied the new-ordinariness of coverings of curves. Under certain assumptions, I solved an open problem posed by Tamagawa concerning the existence of Raynaud-Tamagawa theta divisors for generic curves. Moreover, by applying the theory of theta divisors, I found a sufficient and necessary condition for generic curves whose cyclic tame coverings are new-ordinary.

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5．主な発表論文等

〔雑誌論文〕　計4件（うち査読付論文　4件／うち国際共著　0件／うちオープンアクセス　0件）

1．著者名 Yu Yang	4．巻 2023
2．論文標題 On topological and combinatorial structures of pointed stable curves over algebraically closed fields of positive characteristic	5．発行年 2023年
3．雑誌名 Mathematische Nachrichten	6．最初と最後の頁 1－42
掲載論文のDOI（デジタルオブジェクト識別子） 10.1002/mana.202100053	査読の有無 有
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1．著者名 Yu Yang	4．巻 587
2．論文標題 Raynaud-Tamagawa theta divisors and new-ordinariness of ramified coverings of curves	5．発行年 2021年
3．雑誌名 J. Algebra	6．最初と最後の頁 263-294
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.jalgebra.2021.07.032	査読の有無 有
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1．著者名 Yang Yu	4．巻 295
2．論文標題 On the averages of generalized Hasse?Witt invariants of pointed stable curves in positive characteristic	5．発行年 2020年
3．雑誌名 Mathematische Zeitschrift	6．最初と最後の頁 1～45
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00209-019-02329-2	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

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4．発表年 2023年

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2021年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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6．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

7．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

8．本研究に関連して実施した国際共同研究の実施状況

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