【Grant-in-Aid for Scientific Research (S)】

A new development of arithmetic geometry by p-adic methods

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Project Information	Project Number : 24H00015 Keywords : slope jump, Frobenius map, p-adic cohomology, rigid analytic geometry, arithmetic geometry	Project Period (FY) : 2024-2028

Purpose and Background of the Research

●Outline of the Research

Arithmetic geometry is a research field which treats figures defined by algebraic equations over the rational integer ring $\mathbb{Z}$ (called arithmetic varieties) such as

$$x^n + y^n = z^n \quad (n \geq 3)$$

in Fermat's last theorem. One investigates numbers of rational points, Galois representations, periods and zeta functions associated to arithmetic varieties. For a prime number p, the p-adic method is a way of studying algebraic equations by using modulo p and its lifts. The p-adic methods have been developed in the last 20 years by many mathematicians including our group. We attempt to solve various problems in arithmetic geometry that have become accessible by utilizing p-adic methods.

Legendre's family of elliptic curves / $\mathbb{Q}$

$X: \; y^2 z = x(z-x)(z-tx)$

$\downarrow$

$C = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$: t-variable

mod p $\Rightarrow$ reduction

Legendre's family / $\mathbb{F}_p$

$X(p)\!: y^2 z = x(z-x)(z-tx)$

$\downarrow$

$C(p) = \mathbb{P}^1_{\mathbb{F}_p} \setminus \{0, 1, \infty\}$

$\Downarrow$ period integral

$\Downarrow$ p-adic cohomology

Gaussian hypergeometric function

$$F(t) = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-tx)}}$$

p-adic $\Rightarrow$ analytification

Frobenius on Gaussian hypergeometric differential equation

(F-isocrystals)

Theorem. For Taylor's expansion of above $F(t)$ at $t = \alpha \in \mathbb{Q} \setminus \{0, 1\}$

$$F(t) = \sum_{n=0}^{\infty} a_n (t-\alpha)^n \quad (a_n \in \mathbb{Q}),$$

the growth of exponents of p-power in denominator for sequence a_n (p-adic logarithmic growth) depends only on whether the mod p fiber $X_\alpha(p)$ at $t = \alpha$ is ordinary or supersingular (Figure 2).

Figure 1. Example of p-adic phenomena (p: a prime number)

●Background of the Research

In the late 1960s Igusa found the monodromy of p-adic representations associated to a family of elliptic curves in characteristic p was huge at supersingular points. PI showed the analogous phenomena always happened at jumping points of slopes on a family. Moreover, PI affirmatively solved Kedlaya's "Minimal slope conjecture" which implies the family of elliptic curves is recovered from the huge representation. However, we do not understand essential meanings. We expect to understand mysterious p-adic phenomena such as slope jumps by utilizing p-adic methods.

●Purpose of the Research

The purpose of the research is to study arithmetic geometry by using p-adic methods. In particular we investigate the following problems:

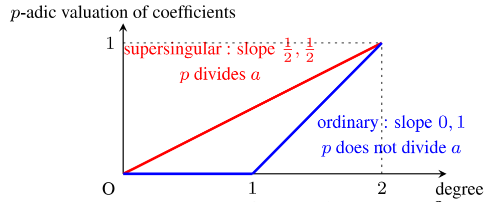
A. Study of p-adic phenomena under the keyword "slope jumps" and its applications
B. p-adic geometricity of arithmetic differential equations and G-functions
C. Further studies of p-adic cohomology theory

E : an elliptic curve / $\mathbb{F}_p$

$$a = 1 + p - \#\, E(\mathbb{F}_p)$$

Zeta function of E

$$Z(E,t) = \frac{1 - at + pt^2}{(1-t)(1-pt)}$$



Newton polygon of $1 - at + pt^2$

Figure 2. Slopes of elliptic curves over a finite field

Expected Research Achievements

●Expected Achievements of the Research

・Understanding of essential meaning of phenomena occurring by slope jumps
・Creating approach methods for problems over the rational field $\mathbb{Q}$
・Developments of p-adic methods for arithmetic and algebraic geometries

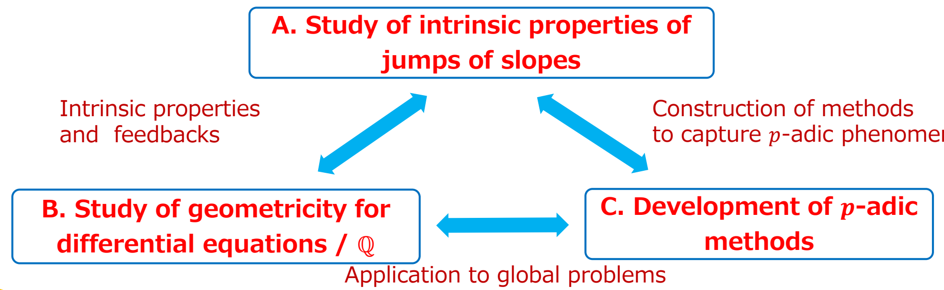


Figure 3. Relation of researches on Igusa's phenomena

●Originality of the Research

In arithmetic geometry there are two methods, ℓ-adic and p-adic, which are equivalent to each other. However, in order to capture slopes and their jumps, the p-adic method is natural comparing to the ℓ-adic method. We investigate geometric and arithmetic meaning of phenomena around jumping points of slopes starting from the minimal slope conjecture which was solved by PI (Problem A). The new findings are possibly obtained if we develop p-adic cohomological methods (Problem C) further. We also investigate problems over the rational number field $\mathbb{Q}$ (Problem B). This research is our original which is based on the studies of our groups.

●Influence of the Research

・Applications to and feedbacks from number theory and arithmetic geometry
・Becoming importance of studies over the rational integer $\mathbb{Z}$ increasingly in algebraic geometry and mathematical physics
・Understanding of mathematical phenomena by p-adic methods --- e.g., "Non-existence of non-isotrivial family of curves over Abelian varieties" by PI (2021)

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