【Grant-in-Aid for Scientific Research (S)】

Analysis of critical interactions in wave fields

Principal Investigator	Waseda University, Faculty of Science and Engineering, Professor OZAWA Tohru	Researcher Number : 70204196
Project Information	Project Number : 24H00024	Project Period (FY) : 2024-2028
	Keywords : Wave Fields, Critical Interactions, Modified Energy, Hamilton Structure	

Purpose and Background of the Research

●Outline of the Research

① Wave Field as a Research Subject

Wave field is nonlinear wave arising in the classical field theory such as matter field, electromagnetic field, gravity field, and so on. Among others, (real and complex) scalar fields and gauge fields describing the nonlinear Klein-Gordon field, the nonlinear Schrödinger field, the Nonlinear Dirac field, and the Maxwell-Schrödinger field are representative examples and are recognized mathematically as solutions for the nonlinear Klein-Gordon equation, the nonlinear Schrödinger equation, and the nonlinear Dirac equation with self-interaction, and for the Maxwell-Schrödinger system as an interacting system of scalar and gauge fields, respectively. Those solutions are often called wavefunctions and are represented by scalar- or vector-valued functions of independent valuables in space-time.

The quantum field theory has a close similarity with the classical field theory and is in fact based on the mathematical foundation of the classical field theory.

Nonlinear partial differential equations Arising in the Classical Field Theory

Hyperbolic Type:	Nonlinear Wave Equation	$(\partial_t^2 - \Delta)\varphi = \Box\varphi = f(\varphi)$
	Nonlinear Klein-Gordon Equation	$\Box\varphi + \varphi = f(\varphi)$
Dispersive Type:	Nonlinear Schrödinger Equation	$i\partial_t\varphi + \Delta\varphi = f(\varphi)$
Elliptical Type:	Equation of Standing Wave	$-\Delta\varphi + \lambda\varphi = f(\varphi)$

Here φ is a real- or complex-valued function on the $(1+n)$ -dimensional space-time, Δ is n-dimensional Laplacian, f stands for self-interaction, and $\lambda > 0$.

Figure 1. Nonlinear partial differential equations Arising in the Classical Field Theory

② History of the Mathematical Research on Wave Field

It was the early 1960s when the simplest models such as nonlinear Klein-Gondor and wave equations were studied in the framework of modern mathematical analysis for the first time as in the pioneer works of Jörgens and Segal. 70 years have passed by since then and we now have a deep and detailed theoretical system of mathematical analysis on scalar fields such as the nonlinear Klein-Gordon, wave, and Schrödinger fields. It is a remarkable historical fact that, in the process of the development, the study on wave field enhanced related subjects such as functional analysis, harmonic analysis, geometric analysis, real analysis, stochastic analysis, and so on. In fact, we see the title "Non-linear Semi-groups" of the pioneering paper by Segal and the Fields Medals for Bourgain (1994) and Tao (2006) for outstanding contribution to this field by introducing methods of harmonic analysis, number theory, probability, and combinatorics as symbolic events of the study of wave fields interacting other subjects. We sometimes call the study of wave fields as "Nonlinear Wave Equations." It means the study on nonlinear hyperbolic and dispersive equations in a narrow sense. However, if the stability analysis of standing waves is a natural subject of the study, then the study of "Nonlinear Wave Equations" includes elliptic equations and associated variational problems. Moreover, the study of "Nonlinear Wave Equations" includes specific subjects in the theory function spaces, harmonic analysis, and geometric analysis such as the embedding theory and the boundedness of oscillatory integrals. Given this perspective, the subject "Nonlinear Wave Equations" has already formed a broad field in the mathematical analysis with large number of researchers worldwide and is on an emerging state.

③ Latest Global Trend of the Research

In the past quarter of the century, many major open problems such as the scattering problem in the energy critical case and the identification problem of blow-up rate in the mass critical case have been solved and the associated theories are systemized at the highest level. The latest global trend of the research in this field is twofold. One is to application of the theories to various equations. The other is to refinement of the theories to specific equations. Innovation is expected in the perspective in this field.

④ Purpose of the Research

The purpose of this research is to establish a new perspective in the study of "Nonlinear Wave Equations" from the point of view of the following three directions:

(i) Refined Structure of Null Structure of Critical Interactions via Reduced System of Equations
(ii) Variation Structure of Modified Energies via Time-Translation Symmetry
(iii) Hamilton Structure via Boundedness and Completeness

For that purpose, we pursue a systematic study by collaboration of three methods by (a) Asymptotic Analysis, (b) Harmonic and Geometric Analysis, (c) Variational Analysis.

●Research Plans

① Improvement of the Research Environment

We are planning to improve the research environment of the laboratory of mathematical physics at Waseda University to carry out joint-works effectively and to check the progress and explore development in the meeting of our research group.

② Organization of the International Workshops

- Harmonic Analysis and Nonlinear Partial Differential Equations, RIMS, Kyoto University
- International Workshop on Fundamental Problems in Mathematical and Theoretical Physics, Waseda University
- Sapporo Symposium on Partial Differential Equations, Hokkaido University
- Nonlinear PDE Session, ISAAC Congress

③ Cooperation with University of Pisa

- Research Promotion of Joint-Works
- Joint-Program for Graduate Education under the Cotutelle Agreement

Expected Research Achievements

We make an emphasis on three viewpoints in item ④ in Purpose and Background of the Research and conduct the research plans with the following three subjects:

(i) Refined Structure of Null Structure of Critical Interactions via Reduced System of Equations

We apply the method of derivation of reduced system of equations based on the asymptotic analysis in high frequencies to various hyperbolic systens and try to make a clear picture of the method in connection with null structure and large time behavior of wavefunctions.

(ii) Variation Structure of Modified Energies via Time-Translation Symmetry

We give a characterization of renormalized terms in the modified higher energy on the basis of the time derivative and make a clear account on the interaction between linear and nolinear terms.

Moreover, we try to characterize the modified higher energy in terms of variation formula of higher order.

(iii) Hamilton Structure via Boundedness and Completeness

We reformulate the compactness argument on the existence of weak solutions and decouple it into the boundedness of approximate solutions and its convergence in a weaker topology. Moreover, we reexamine Kato's general theory for equations with Hamilton structure and try to make a different approach on the basis of boundedness and completeness instead of compactness.

Homepage Address, etc.	http://www.ozawa.phys.waseda.ac.jp/index.html