様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

科研費 KAKENHI

# 科学研究費助成事業　　研究成果報告書

平成 ３０ 年　 ６ 月 １５ 日現在

機関番号：３２６８９
研究種目：基盤研究(A)（一般）
研究期間：2013～2017
課題番号：２５２４７００５

研究課題名（和文）Systematic development and application of methods in differential geometry and integrable systems motivated by quantum cohomology

研究課題名（英文）Systematic development and application of methods in differential geometry and integrable systems motivated by quantum cohomology

研究代表者
Ｇｕｅｓｔ　Ｍａｒｔｉｎ（Guest, Martin)

早稲田大学・理工学術院・教授

研究者番号：１０２９５４７０

交付決定額（研究期間全体）：（直接経費）　　22,300,000円

研究成果の概要（和文）：当研究ではtt*-戸田方程式の解法をいくつか開発した。偏微分方程式論を用いる解法、可積分系理論を用いる解法、そしてリー理論を用いる解法である。主結果はA_n型tt*-戸田方程式に対して得られ、解とその漸近データ、モノドロミー・データを完全に与えた。また、n=1の場合では更に抽象的なアプローチを用いて、解のモジュライ空間を記述した。以上の結果はいずれも、ある種のケーラー多様体ないし軌道体の量子コホモロジーに対応する特殊解が動機の一つである。この分野の研究促進のため、専門家による研究集会およびセミナーを数多く企画した。

研究成果の概要（英文）：A series of methods for solving the tt*-Toda equations were developed during the course of this project. These methods used p.d.e. theory, integrable systems theory, and Lie theory. Our main results were achieved for the tt*-Toda equations of type A_n.  Here we give a complete treatment of the solutions and their asymptotic data and monodromy data. A more abstract approach was used in the case n=1, in order to describe the moduli space of solutions. These results were motivated in part by the special solutions corresponding to quantum cohomology rings of Kaehler manifolds. In order to promote research in this area, a number of conferences, workshops, and seminars by specialists were organised.

研究分野： 数物系科学，微分幾何学

キーワード： Integrable systems　Geometry　Quantum cohomology

## １．研究開始当初の背景

Ideas from differential geometry and integrable systems (especially tools and ideas from infinite-dimensional Lie theory) have been used to study harmonic maps since the 1990's. This area is well-developed and there are many applications. On the other hand these methods are mainly local, and global results are few; compact target spaces have been well-studied, but noncompact target spaces are more difficult. We have proposed a new source of ideas and examples: the area surrounding quantum cohomology, Frobenius manifolds, and mirror symmetry.

Quantum cohomology is deeply related to the theory of integrable systems. Dubrovin [D1] showed that the WDVV equations (Witten-Dijkgraaf-Verlinde-Verlinde) can be formulated as an isomonodromy problem. Cecotti and Vafa [CV] introduced the tt* equations (topological - antitopological fusion equations), and Dubrovin [D2] showed that these equations have a zero curvature formulation - indeed they are a special case of the equations for pluriharmonic maps into GL(n,R)/O(n).

## ２．研究の目的

As explained above, the tt* equations have rich significance in mathematical physics and geometry. They are integrable in the same sense as harmonic map theory and also in the sense of isomonodromy theory. The existence of remarkable global solutions was conjectured by physicists, but few concrete examples were known. We had already proved existence in some cases, and the aim of this project was to develop systematic methods, focusing on the case of the tt*-Toda equations.

## ３．研究の方法

Initially our methods were based on two themes, both related to the tt*-Toda equations of type A_n. Both themes are based on the fact that solutions of tt*-Toda equations correspond to three kinds of data: holomorphic data (loop group theory), asymptotic data (p.d.e. theory), and monodromy data (isomonodromy theory).

Theme 1: extension and interpretation of previous results (such as [GIL1]) on the tt*-Toda equations

Here we used a combination of methods from p.d.e. theory, isomonodromy theory, and loop group theory. Although the tt*-Toda equations are nonlinear p.d.e., it is possible to use elementary methods effectively to study the solutions and their symptotics. In particular, the shape of the tt*-Toda equation is suitable for applying the maximum principle. The main difficulty is the fact that we have system of equations, rather than a scalar equation. This is also an obstacle to applying standard isomonodromy theory, but fortunately the tt*-Toda equations have many symmetries and these lead to a simplification of the method of computing the monodromy data.

Theme 2: intrinsic approach via harmonic bundles and TERP structures.

In Theme 1 we used classical methods and focused on explicit computations. However, for a deeper understanding, an intrinsic "matrix independent" approach is needed. We used such an approach, based on the theory of harmonic bundles and TERP structures, based on [H].
TERP structures are harmonic bundles with extra conditions; they are solutions of a coordinate-free version of the tt* equations. They generalize the Dubrovin connection in quantum cohomology theory, and have an isomonodromic deformation interpretation. Initially we focused on the very simplest case of the tt*-Toda equations of type A_1, which is essentially the Third Painleve equation. Nevertheless, this case is sufficiently nontrivial to reveal new features.

## ４．研究成果

Theme 1:
In joint work with C.-S. Lin we had established the existence of some globally smooth solutions in the case n=3, and in [GIL1] we extended this to all globally smooth solutions in the case n=3. We also calculated the Stokes data corresponding to these solutions. This computation of monodromy data is the "direct monodromy problem".

In [GIL2] we studied the "inverse monodromy problem" for the tt*-Toda equations. This amounted to a direct attack on the Riemann-Hilbert problem, which we have attacked indirectly by p.d.e. methods in [GIL1]. In some ways we found that the Riemann-Hilbert approach is stronger than the p.d.e. approach, e.g. it gives more explicit asymptotic formulae for solutions, and it gives information on solutions which are assumed to be smooth only near infinity. In other ways the Riemann-Hilbert approach is weaker, as it applies only to a subset of the globally smooth solutions.

In [GIL3] we used the loop group Iwasawa factorization method (DPW method) to study a larger set of solutions, namely the solutions which are smooth near zero. This method makes use of the holomorphic data, which is closer (than the asymptotic or monodromy data) to the geometry and physics of the solutions. There are two cases: (i) generic holomorphic data, (ii) non-generic holomorphic data. In the initial part of the project we completed quite quickly the calculations for (i). The calculations for (ii) presented unexpected difficulties, which took longer to overcome. However, as a result, we now have a very complete description of the holomorphic data, asymptotic data, and monodromy data when n=3. Our methods appear to be sufficient to deal with general n as well; this is a future task.

We were able to generalize some of these results to the tt*-Toda equations of type A_n for general n in [GH1], and further to the case of any Lie algebra type in [GH2]. The fundamental new ingredient here was the Lie-theoretic treatment of Stokes data developed by P. Boalch in [B]. This made possible a more systematic description of the monodromy data and a simplification of the calculations. An unexpected development arising from these calculations was a new link with some classical results on Lie theory by B. Kostant and R. Steinberg. These were also published in [GH2].

Theme 2:

The lengthy article [GH] was published towards the end of the project period. This contains a more intrinsic (coordinate-free) approach to the tt*-Toda equations in the case A_1. Although it is based in part on explicit calculations similar to those in Theme 1, it uses a different language, which permits new applications. The main new tool is the construction of a (canonical) moduli space of bundles with meromorphic connections, and (non-canonical) identifications with the space of asymptotic data, and with the space of monodromy data. The main new result is a complete description of the zeros and poles of (not necessarily globally smooth) solutions of the tt*-Toda equations in this case. In particular we showed that there are exactly 14 types of behaviour.

The budget for this project was used mainly for travel expenses of researchers related to the project, including partial support for several conferences, workshops, and invited lectures. The main events were as follows:

Conference on "Differential Geometry and Differential Equations: the influence of Mirror Symmetry and Physics", December 2017, Waseda University (speakers included: P. Boalch, S. Heller, K. Iwaki, C.-S. Lin, I. McIntosh, R. Miyaoka, H. Nagoya)

"1st Japan-Taiwan Conference on Differential Geometry", December 2016, Waseda University

"String Theory Meeting in the Greater Tokyo Area", November 2016, Waseda and Tokyo Metropolitan University, (speakers included: R. Bielawski, T. Eguchi, S. Hellerman, M. Jimbo, O. Lisovyy)

Workshop on "Flat connections, Higgs bundles and Painleve equations", May 2016, National Taiwan University (speakers included: P. Boalch, C. Hertling, A. Its, Y.-P. Lee, T. Mochizuki, C. Sabbah, M.-H. Saito)

Workshop on "Painleve equations and related topics", May 2015, National Taiwan University (speakers included: H. Chiba, T. Mano, H. Nagoya, Y. Ohyama, M.-H. Saito)

Joint International Workshop on "Integrable Systems and Mathematical Physics", March 2014, National Taiwan University (speakers included: E. Basor, O. Costin, B. Dubrovin, O. Lisovyy, M. Noumi, M. Okado, M.-H. Saito)

The following smaller events were also organised, all at Waseda University:

Lectures on Quasi-Hamiltonian Geometry by Prof. Eckhard Meinrenken (University of Toronto), 23-24 June 2017

Germany-Japan One-Day Workshop on Geometry and Topology, 22 May 2017

Workshop on integrable and nonintegrable lattice models: theory and computation, 15-16 July 2016

Workshop on Geometry and Nonlinear PDE, 13 January 2015

Moduli spaces of flat connections on surfaces and related topics, 15-16 November 2014

Symplectic geometry of moduli spaces of connections, 14 February 2014

Isomonodromic deformations and related topics, 22-23 November 2013

Workshop on integrable systems, modular forms, and related applications, 31 May 2013 and 7 June 2013

A series of study meetings "Koriyama Geometry and Physics Days" was held at Nihon University (Koriyama, Fukushima) as follows: "Noncommutative geometry and related topics" (February 2018), "Geometric quantization and related topics" (February 2017), "Painleve equations, integrable systems and moduli spaces" (February 2016), "String topology, orbifolds, and related topics" (October 2014)

文献

[B] P. Boalch, G-Bundles, isomonodromy, and quantum Weyl groups, Int. Math. Res. Notices 2002 (2002), 1129--1166
[CV] S.Cecotti, C. Vafa, Topological-anti-topological fusion, Nuclear Phys. B 367 (1991) 359-461
[D1] B. Dubrovin, Geometry of 2D topological field theories, Lecture Notes in Math. 1620, Springer, 1996, 120-348
[D2] B. Dubrovin, Geometry and integrability of topological-antitopologicalfusion, Comm. Math. Phys. 152, (1993) 539-564
[GH] M. A. Guest, C. Hertling, Painleve III: A Case Study in the Geometry of Meromorphic Connections, Lecture Notes in Mathematics 2198, Springer, 2017 (202 pages)
[GH1] M. A. Guest, N.-K. Ho, A Lie-theoretic description of the solution space of the tt*-Toda equations, Math. Phys. Anal. Geom., published online 2017 (27 pages)
[GH2] M. A. Guest, N.-K. Ho, Kostant, Steinberg, and the Stokes matrices of the tt*-Toda equations, preprint (arXiv: 1802.01126) (42 pages)
[GIL1] M. A. Guest, A. R. Its, and C.-S. Lin, Isomonodromy aspects of the tt* equations of Cecotti and Vafa I. Stokes data, Int. Math. Res. Notices 2015 (2015) 11745-11784
[GIL2] M. A. Guest, A. R. Its, and C.-S. Lin, Isomonodromy aspects of the tt* equations of Cecotti and Vafa II. Riemann-Hilbert problem, Commun. Math. Phys. 336 (2015) 337-380
[GIL3] M. A. Guest, A. R. Its, and C.-S. Lin, Isomonodromy aspects of the tt* equations of Cecotti and Vafa III. Iwasawa factorization and asymptotics, preprint (arXiv:1707.00259) (59 pages)
[H] C. Hertling, tt* geometry, Frobenius manifolds, their connections, and the construction for singularities", J. reine angew. Math. 555 (2003) 77-161

５．主な発表論文等
（研究代表者、研究分担者及び連携研究者には下線）

〔雑誌論文〕（計　34　件）

1. Aldabergenov, Yermek; Ketov, Sergei V.; Removing instability of inflation in Polonyi-Starobinsky supergravity by adding FI term. Modern Phys. Lett. A, 査読有, 33, 2018, 8pp. 10.1142/S0217732318500323
2. Guest, Martin A.; Ho, Nan-Kuo, A Lie-theoretic description of the solution space of the tt*-Toda equations. Math. Phys. Anal. Geom. 査読有, 20, 2017, 27pp. 10.1007/s11040-017-9255-z
3. Guest, Martin A.; Hertling, Claus, Painleve III: a case study in the geometry of meromorphic connections. Lecture Notes in Mathematics, 査読有 2198, 2017, 202pp. 10.1007/978-3-319-66526-9
4. Ohnita, Yoshihiro, On Floer homology of the Gauss images of isoparametric hypersurfaces. Springer Proc. Math. Stat. 査読有, 203, 2017, 235-247, 10.1007/978-981-10-5556-0_20
5. Ohno, Shinji; Sakai, Takashi; Urakawa, Hajime, Biharmonic homogeneous submanifolds in compact symmetric spaces. Springer Proc. Math. Stat. 査読有, 203, 2017, 323-333, 10.1007/978-981-10-5556-0_27
6. Kobayashi, Shimpei, Nonlinear d'Alembert formula for discrete pseudospherical surfaces. J. Geom. Phys. 査読有, 119, 2017, 208-223,

(ア) 10.1016/j.geomphys.2017.05.001

7. Iritani, Hiroshi, A mirror construction for the big equivariant quantum cohomology of toric manifolds. Math. Ann. 査読有, 368, 2017, 279-316, 10.1007/s00208-016-1437-7
8. Hosono, Shinobu; Takagi, Hiromichi, Towards homological projective duality for Sˆ2Pˆ3 and Sˆ2Pˆ4 Adv. Math. 査読有, 317, 2017, 371-409, 10.1016/j.aim.2017.06.039
9. Iriyeh, Hiroshi; Ma, Hui; Miyaoka, Reiko; Ohnita, Yoshihiro, Hamiltonian non-displaceability of Gauss images of isoparametric hypersurfaces. Bull. Lond. Math. Soc. 査読有, 48, 2016, 802-812, doi.org/10.1112/blms/bdw040
10. Alexandrov, Sergei; Ketov, Sergei V.; Wakimoto, Yuki, Non-perturbative scalar potential inspired by type IIA strings on rigid CY. J. High Energy Phys. 査読有, 66, 2016, 38pp. doi.org/10.1007/JHEP11(2016)066
11. Dorfmeister, Josef F.; Inoguchi, Jun-Ichi; Kobayashi, Shimpei, A loop group method for minimal surfaces in the three-dimensional Heisenberg group. Asian J. Math. 査読有, 20, 2016, 409-448, 10.4310/AJM.2016.v20.n3.a2
12. Dorfmeister, Josef F.; Inoguchi, Jun-ichi; Kobayashi, Shimpei, A loop group method for affine harmonic maps into Lie groups. Adv. Math. 査読有, 298, 2016, 207-253, 10.1016/j.aim.2016.04.018
13. Iritani, Hiroshi; Xiao, Jifu, Extremal transition and quantum cohomology: examples of toric degeneration. Kyoto J. Math. 査読有, 56, 2016, 873-905, 10.1215/21562261-3664959
14. Galkin, Sergey; Golyshev, Vasily; Iritani, Hiroshi, Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures. Duke Math. J. 査読有, 165, 2016, 2005-2077, doi:10.1215/00127094-3476593
15. Hosono, Shinobu; Takagi, Hiromichi, Double quintic symmetroids, Reye congruences, and their derived equivalence, J. Differential Geom, 査読有, 104, 2016, 443-497, 10.4310/jdg/1478138549
16. Ma, Hui; Ohnita, Yoshihiro, Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces II. Tohoku Math. J. 査読有, 67, 2015, 195-246
17. Bieliavsky, Pierre; de Goursac, Axel; Maeda, Yoshiaki; Spinnler, Florian, Non-formal star-exponential on contracted one-sheeted hyperboloids, Adv. Math. 査読有, 291, 2016, 362-402, 0.1016/j.aim.2015.12.026
18. Maeda, Yoshiaki; Rosenberg, Steven; Torres-Ardila, Fabian, The geometry of loop spaces II: characteristic classes, Adv. Math. 査読有, 287, 2016, 485-518, 10.1016/j.aim.2015.10.001
19. Ohno, Shinji; Sakai, Takashi; Urakawa, Hajime, Biharmonic homogeneous hypersurfaces in compact symmetric spaces, Differential Geom. Appl, 査読有, 43, 2015, 155-179, 10.1016/j.difgeo.2015.09.005
20. Kobayashi, Shimpei, A loop group method for Demoulin surfaces in the 3-dimensional real projective space, Differential Geom. Appl, 査読有, 40, 2015, 57-66, 10.1016/j.difgeo.2015.02.005
21. Coates, Tom; Iritani, Hiroshi, On the convergence of Gromov-Witten potentials and Givental's formula, Michigan Math. J, 査読有, 64, 2015, 587-631, doi:10.1307/mmj/1441116660
22. Iritani, Hiroshi; Milanov, Todor; Tonita, Valentin, Reconstruction and convergence in quantum K-theory via difference equations, Int. Math. Res. Not, 査読有, 11, 2015, 2887-2937, 10.1093/imrn/rnu026
23. Hosono, Shinobu, Moduli spaces of Calabi-Yau complete intersections, Nuclear Phys. B, 査読有, 898, 2015, 661-666,10.1016/j.nuclphysb.2015.04.001
24. M. Guest, A, R. Its, C.-S. Lin, Isomonodromy aspects of the tt* equations of Cecotti and Vafa I. Stokes data, Int. Math. Res. Notices, 査読有, 2014, 10.1093/imrn/rnu250
25. M. Guest, A. R. Its, C.-S. Lin, Isomonodromy aspects of the tt* equations of Cecotti and Vafa II. Riemann-Hilbert problem, Commun. Math. Phys. 336, 査読有, 2015 337-380, 10.1007/s00220-014-2280-x
26. M. Guest、A, H. Sakai, Orbifold quantum D-modules associated to weighted projective spaces, Comm. Math. Helv. 89, 査読有, 2014, 273-297, 10.4171/CMH/319
27. M. Guest A, C.-S. Lin, Nonlinear PDE aspects of the tt* equations of Cecotti and Vafa, J. reine angew. Math. 689, 査読有, 2014 1-32, 10.1515/crelle-2012-0057
28. Ma, Hui; Ohnita, Yoshihiro, Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces. I, J. Differ. Geom, 97, 査読有, 2014, 275-348, 10.2748/tmj/1435237041
29. Ketov, Sergei; Watanabe, Natsuki, On the Higgs-like quintessence for dark energy, Mod. Phys. Lett. A 29, Article ID 1450117, 査読有, 2014, 10.1142/S021773231450117X

30. Dorfmeister, Josef; Inoguchi, Jun-ichi; Kobayashi, Shimpei, Constant mean curvature surfaces in hyperbolic 3-space via loop groups, J. Reine Angew. Math. 686, 査読有, 2014, 1-36, 10.1515/crelle-2012-0024
31. Chiodo, Alessandro; Iritani, Hiroshi; Ruan, Yongbin, Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence, Publ. Math., Inst. Hautes Etud. Sci. 119, 査読有, 2014, 127-216, 10.1007/s10240-013-0056-z
32. Hosono, Shinobu; Takagi, Hiromichi, Determinantal quintics and mirror symmetry of Reye congruences, Commun. Math. Phys. 329, 査読有, 2014, 1171-1218, 10.1007/s00220-014-1971-7
33. Hosono, Shinobu; Takagi, Hiromichi, Mirror symmetry and projective geometry of Reye congruences I, J. Algebr. Geom. 23, 査読有, 2014, 279-312, 10.1090/S1056-3911-2013-00618-9
34. Iriyeh Hiroshi; Sakai Takashi; Tasaki Hiroyuki, Lagrangian Floer homology of a pair of real forms in Hermitian symmetric spaces of compact type, J. Math. Soc. Japan, 65, 査読有, 2013, 1135-1151, 10.2969/jmsj/06541135

〔学会発表〕(計 12 件)
1. Martin Guest, Some applications of irregular meromorphic o.d.e. to geometry, Taiwan Geometry Symposium, 2018
2. Martin Guest, A general theory of tt*-Toda equations, Algebraic Geometry and Symplectic Geometry at Crossroads, 2017
3. Guest Martin, Patterns in Stokes data of the radial Toda equations, Mathematical Physics seminar, 2017
4. Martin Guest, On the monodromy data of the tt*-Toda equations, Hodge Theory, Stokes Phenomenon and Applications, 2017
5. Martin Guest, The enhanced Coxeter Plane - an application of integrable systems to Lie groups, Differential Geometry seminar, 2017
6. Martin Guest, Convexity for a certain space of solutions to the Hitchin equations, LMS-EPSRC Durham Symposium on Geometric and Algebraic Aspects of Integrability, 2016
7. Martin Guest, Harmonic maps of Painlevé-type: the loop group point of view, Workshop on Flat connections, Higgs bundles and Painleve equations, 2016
8. Martin Guest, Painleve III: the geometry of the solutions, Workshop on Geometry and Nonlinear PDE, 2015
9. Martin Guest, Classification of N=2 supersymmetric theories, revisited, Workshop: Special Geometric Structures in Mathematics and Physics, 2014
10. Martin Guest, The tt*-Toda equations, Pacific Rim Geometry Conference, 2013
11. Martin Guest, Solving integrable systems: old and new viewpoints, Symplectic geometry and mathematical physics seminar, 2013
12. Martin Guest The quantum cohomology of CP^n, from various viewpoints, Differential topology seminar, 2013

〔その他〕ホームページ等
http://www.f.waseda.jp/martin/

6．研究組織
(1)研究代表者
Guest Martin (GUEST, Martin)
早稲田大学・理工学術院・教授
研究者番号：10295470

(2)研究分担者　なし

(3)連携研究者
大仁田　義裕 (OHNITA, Yoshihiro)
大阪市立大学・理学研究科・教授
研究者番号：90183764

前田　吉昭 (MAEDA, Yoshiaki)
東北大学・知の創出センター・副センター長
研究者番号：40101076

Sergei V Ketov (SERGEI, V Ketov)
首都大学東京・理工学研究科・准教授
研究者番号：70347269

酒井　高司 (SAKAI, Takashi)
首都大学東京・理工学研究科・准教授
研究者番号：30381445

乙藤　隆史 (OTOFUJI, Takashi)
日本大学・工学部・准教授
研究者番号：70339266

赤穂　まなぶ (AKAHO, Manabu)
首都大学東京・理工学研究科・准教授
研究者番号：30332935

小林　真平 (KOBAYASHI, Shimpei)
北海道大学・理学研究科・准教授
研究者番号：40408654

入谷　寛 (IRITANI, Hiroshi)
京都大学・理学研究科・教授
研究者番号：20448400

細野　忍 (HOSONO, Shinobu)
学習院大学・理学部・教授
研究者番号：60212198