様　式　Ｃ－１９、Ｆ－１９、Ｚ－１９　（共通）

科学研究費助成事業　　研究成果報告書

平成　２８　年　　６　月　　６　日現在

機関番号：３２６４４
研究種目：基盤研究(C)（一般）
研究期間：2013～2015
課題番号：２５４０００７５

研究課題名（和文）Geometry of geodesics of Riemannian and Finsler manifolds

研究課題名（英文）Geometry of geodesics of Riemannian and Finsler manifolds

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交付決定額（研究期間全体）：（直接経費）　　1,800,000円

研究成果の概要（和文）：１．最小跡はフラクタルとなっているリーマンとフィンスラー構造を構成した（仁一伊藤先生と共同研究）。この結果はフィンスラー幾何学だけではなく、リーマン幾何学でも面白い結果だと思われる。この結果は伊藤・田中定理と矛盾せず、すなわち、我々の構造はなめらかな構造ではない。
２．凸関数をもつフィンスラー多様体の幾何学や位相を研究した（塩濱勝博先生と共同研究）。凸関数をもつリーマン多様体と同じように、凸関数をもつフィンスラー多様体の位相が制限を受けている。さらに、リーマン多様体と違って、フィンスラー計量の測地線はreversibleではないので、その影響を考慮し、リーマンとフィンスラーの違いを明確にした。

研究成果の概要（英文）：1.I have constructed (joint research with J. Itoh) Riemannian and Finslerian structures on spheres whose cut locus of a point is a fractal (i.e. the Hausdorff dimension of the cut locus is not integer). This result is interesting not only for Finsler geometry, but also for Riemannian geometry and it is in the same time consistent with the result of Itoh-Tanaka about the Hausdorff dimension of the cut locus of a smooth Riemannian manifold. Indeed, our Riemannian structure is not a smooth one.
2.I have introduced and studied the notion of convex functions on Finsler manifolds (joint research with K. Shiohama). Similarly with the Riemannian case, we have shown that there are topological restrictions for Finsler manifolds that admit convex functions. The difference with the Riemannian case was also clarified, as well as the influence of non-reversibility of geodesics in the Finslerian setting.As application, I have studied the convexity of Busemann functions on Finsler manifolds.

研究分野： 微分幾何学

キーワード： Riemannian manifolds　Finsler manifolds　geodesics　cut locus　convex functions　Busemann functions　Hausdorff dimension　fractals

1 研究開始当初の背景

1. Finsler geometry is famous for the local computations and the lack of global results.

   When I start this research project, there was almost nothing known about the cut locus of a Finsler manifold. The cut locus is a very important topic in Differential geometry that links the local geometry and the global one. So it was a real need in Finsler geometry for a detailed study of the cut locus.

2. Also, the relation of Finsler geometry with the topology of the base manifold is not clear. By simply assuming that the Finsler manifold admits a convex function, I expected this relation to become clear.

2 研究の目的

In the present research I have planned to study the following problems:

1. The cut locus structure of a Riemannian and Finsler manifold

2. The topological structure, the isometry group and other geometrical properties of a Finsler manifold that admits a convex function.

3 研究の方法

1. I have used methods from the theory of geodesics in order to study the existence of Riemannian structures whose cut locus is a fractal. However, only a small part of these methods are applicable in the Finslerian case. So new geometrical techniques need to be developed for the Finsler case.

2. The topological structure and other geometrical properties of Finsler manifolds admiting a convex function cannot be studied using classical tools used in Riemannian geometry, as Rauch comparison theorem or Toponogov comparison. Therefore, I will use only variational formulas for Finsler manifolds and the properties of the Busemann function.

4 研究成果

1. The cut locus.

   (a) Finslerian distance and cut locus structure ④

      Theorem 4.1 *Let N be a closed subset of a backward complete arbitrary dimensional Finsler manifold (M, F). Then, the distance function d_N from the subset N is differentiable at a point $q \in M \setminus N$ if and only if q admits a unique N-segment.*

      The following theorems are the main theorems about the cut locus. Their counterparts for the cut locus of a compact subset of an Alexandrov surface have been proved by Shiohama and Tanaka. However, we point out that the key tool for proving these in the Riemannian or Alexandrov spaces case was the Toponogov comparison theorem, that does not hold for Finsler manifolds. Hence, we gave completely different proofs to these theorems.

      Theorem 4.2 *Let N be a closed subset of a backward complete 2-dimensional Finsler manifold*

(M, F). *Then, the cut locus* C_N *of* N *satisfies the following properties:*

i. C_N *is a local tree and any two cut points on the same connected component of* C_N *can be joined by a rectifiable curve in* C_N.

ii. The topology of C_N *induced from the intrinsic metric* δ *(see Section 7 for definition) coincides with the topology induced from* (M, F).

iii. The space C_N *with the intrinsic metric* δ *is backward complete, provided* $\inf_{q \in N} d_N(C_N, q) > 0$.

iv. The cut locus C_N *is a union of countably many Jordan arcs except for the endpoints of* C_N.

Theorem 4.3 *There exists a set* $\mathcal{E} \subset [0, \sup d_N)$ *of measure zero with the following properties:*

i. For each $t \in (0, \sup d_N) \setminus \mathcal{E}$, *the set* $d_N^{-1}(t)$ *consists of locally finitely many mutually disjoint arcs. In particular, if* N *is compact, then* $d_N^{-1}(t)$ *consists of finitely many mutually disjoint circles.*

ii. For each $t \in (0, \sup d_N) \setminus \mathcal{E}$, *any point* $q \in d_N^{-1}(t)$ *admits at most two* N*-segments.*

Remark 4.4 Notice that the cut locus of a closed subset is not always closed, but the space C_N with the intrinsic metric δ is backward complete for any closed subset of a backward complete Finsler surface.

(b) Riemannian and Finsler spheres whose cut locus is a fractal ②

Theorem 4.5 *For any integer* $2 \le k < \infty$ *there is an at least* k*-differentiable Riemannian metric on the* $n(k)$*-dimensional sphere* $\mathbb{S}^{n(k)}$ *and a point* p *in* $\mathbb{S}^{n(k)}$ *such that the Hausdorff dimension of* $\mathcal{C}(p)$ *is a real number between 1 and 2, where* $n(k) := \frac{3^{k-1}+3}{2}$.

Moreover, we show that there is a Finsler metric of Randers type on this sphere with the same property. Indeed, if we use the same notations as in Theorem 4.5, we have

Theorem 4.6 *For any integer* $2 \le k < \infty$, *under the influence of a suitable magnetic field* β *defined on* $\mathbb{S}^{n(k)}$, *there is an at least* k*-differentiable non-Riemannian Finsler metric of Randers type on* $\mathbb{S}^{n(k)}$ *such that the cut locus of the point* p *with respect to this Finsler metric coincides with* $\mathcal{C}(p)$.

(c) The cut locus of a surface of revolution ①

we perturb the induced canonical Riemannian metric h of a surface of revolution by the rotational vector field W obtaining in this way a Randers type metric on M through the Zermelo's navigation process. We study some of the local and global geometrical properties of the geodesics on the surface of revolution M endowed with this Randers metric.

Theorem 4.7 *Let* $(M, F = \alpha + \beta)$ *be the rotational Randers metric constructed from the navigation data* (h, W), *where* (M, h)

is a Riemannian surface of revolution whose warp function is bounded $m(r) < \frac{1}{\mu}$, $\mu > 0$, and $W = \mu\frac{\partial}{\partial\theta}$ is the breeze on M blowing along parallels, then the unit speed Finslerian geodesics $\mathcal{P} : (-\epsilon, \epsilon) \to M$ are given by

$$\mathcal{P}(s) = (r(s), \theta(s) + \mu s), \tag{4.1}$$

where $\gamma(s) = (r(s), \theta(s))$ is a h-unit speed geodesic.

Theorem 4.8 *The rotational Randers space $(M, F = \alpha + \beta)$ can be isometrically embedded into the Minkowski space $(\mathcal{U}_\mu, \tilde{F})$ if and only if the Riemannian surface of revolution (M, h) can be isometrically embedded in $(\mathbb{R}^3, \delta)$.*

The geometry of a Riemannian surface of revolution is completely governed by the Clairaut relation, but the correspondent of this relation in Finsler geometry is unknown. We give here a generalisation of the Riemannian Clairaut relation to the case of a Randers rotational surface of revolution.

Theorem 4.9 *Let $\gamma(s) = (r(s), \theta(s))$ be an h-geodesic of Clairaut constant ν, that makes an angle $\phi(s)$ with the profile curve passing through $\gamma(s)$, and let $\mathcal{P}(s)$ be the corresponding F-geodesic on the Randers rotational surface of revolution (M, F). Then the following relations hold good.*

$$\sqrt{1 + 2\mu\nu + \mu^2 m^2}\cos(\psi - \phi) = 1 + \mu\nu, \tag{4.2}$$

$$m \sin\psi = \frac{\nu + \mu m^2}{\sqrt{1 + 2\mu\nu + \mu^2 m^2}}, \tag{4.3}$$

where ψ is the angle between $\dot{\mathcal{P}}(s)$ and the profile curve passing through $\mathcal{P}(s)$.

Obviously, these two forms of the Clairaut relation are equivalent and they reduce to the classical Clairaut relation when F is Riemannian.

The geometry of geodesics of (M, F) can now be easily obtained using these relations (see Section. We mention here a result about the set of poles of a Randers rotational metric.

Theorem 4.10 *For any point $q \neq p$, let γ be a geodesic from q, which is not tangent to the twisted meridian through q. Then γ cannot be a ray, that is the vertex p is the unique pole of (M, F).*

Concerning the cut locus on the Randers surface, we have proved the following result.

Theorem 4.11 *Let $(M, F = \alpha + \beta)$ be a rotational Randers von Mangoldt surface of revolution. Then, for any point $q \neq p$, the Finslerian cut locus $\mathcal{C}_q^{(F)}$ of q is the Jordan arc*

$$\mathcal{C}_q^{(F)} = \{\varphi(s, \tau_q(s)) : s \in [c, \infty)\},$$

where $\varphi(c, \tau_q(c))$ is the first conjugate point of q along the twisted meridian $\varphi(s, \tau_q(s))$.

2. Finsler manifolds admitting a convex function ③.

A function $\varphi : (M,F) \to \mathbb{R}$ is said to be *convex* if and only if along every (forward and backward) geodesic $\gamma : [a,b] \to (M,F)$, the restriction $\varphi \circ \gamma : [a,b] \to \mathbb{R}$ is convex function, that is:

$$\varphi\circ\gamma((1-\lambda)a+\lambda\, b) \leq (1-\lambda)\varphi\circ\gamma(a)+\lambda\varphi\circ\gamma(b),$$

where $0 \leq \lambda \leq 1$.

Theorem 4.12 *Let $\varphi : (M,F) \to \mathbb{R}$ be a convex function. Assume that all of the levels of φ are compact.*

If $\inf_M \varphi$ is not attained, then there exists a homeomorphism

$$H : M_a^a(\varphi) \times (\inf_M \varphi, \infty) \to M,$$

for an arbitrary fixed number $a \in (\inf_M \varphi, \infty)$, such that

$$\varphi(H(y,t)) = t,\ \forall y \in M_a^a(\varphi),\ \forall t \in (\inf_M \varphi, \infty).$$

Moreover, if $\lambda := \inf_M \varphi$ is attained, then M is homeomorphic to the normal bundle over $M_\lambda^\lambda(\varphi)$ in M.

Next, we discuss the case where φ has a disconnected level.

Theorem 4.13 *Let $\varphi : (M,F) \to \mathbb{R}$ be a convex function. If $M_c^c(\varphi)$ is disconnected for some $c \in \varphi(M)$, we then have*

(1) $\inf_M \varphi$ *is attained.*

(2) *If $\lambda := \inf_M \varphi$, then $M_\lambda^\lambda(\varphi)$ is a totally geodesic smooth hypersurface which is totally convex without boundary.*

(3) *The normal bundle of $M_\lambda^\lambda(\varphi)$ in M is trivial.*

(4) *If $b > \lambda$, then the boundary of the b-sublevel set $M^b(\varphi) := \{x \in M \mid \varphi(x) \leq b\}$ has exactly two components.*

The diameter function $\delta : \varphi(M) \to \mathbb{R}_+$ plays an important role in this article and it is defined as follows:

$$\delta(t) := \sup\{d(x,y) | x, y \in M_t^t(\varphi)\}.$$

It is known that the diameter function δ of a complete Riemannian manifold admitting a convex function is monotone non-decreasing. However it is not certain if it is monotone on a Finsler manifold.

We finally discuss the number of ends of a Finsler manifold (M,F) admitting a convex function φ.

Theorem 4.14 *Let $\varphi : (M,F) \to \mathbb{R}$ be a convex function.*

(a) Assume that φ admits a disconnected level.

  (A1) *If all the level of φ are compact, then M has two ends.*

  (A2) *If all the levels of φ are non-compact, then M has one end.*

  (A3) *If both compact and non-compact levels of φ exist simultaneously, then M has at least three ends.*

(b) Assume that all the levels of φ are connected and compact.

  (B1) *If $\inf_M \varphi$ is attained, then M has one end.*

  (B2) *If $\inf_M \varphi$ is not attained, then M has two ends.*

(c) If all the levels are connected and non-compact, then M has one end.

(d) Assume that all the levels of φ are connected and that φ admits both compact and non-compact levels simultaneously. Then we have:

(D1) *If* $\inf_M \varphi$ *is not attained, then* M *has two ends.*

(D2) *If* $\inf_M \varphi$ *is attained, then* M *has at least two ends.*

(e) *Finally, if* M *has two ends, then all the levels of* φ *are compact.*

5　主な発表論文等

〔雑誌論文〕（計　11　件）

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（他９件）

〔図書〕（計　0　件）
〔産業財産権〕○出願状況（計　0　件）
名称：発明者：権利者：種類：番号：出願年月日：国内外の別：
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名称：発明者：権利者：種類：番号：取得年月日：国内外の別：
〔その他〕ホームページ等

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