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研究課題名（和文）W-constraints and the Eynard-Orantin topological recursion

研究課題名（英文）W-constraints and the Eynard-Orantin topological recursion

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研究成果の概要（和文）：私の研究の主なテーマは，ある種のGromov-Witten型の不変量を再構成することである.この問題は，スクリーニング作用素を用いて定義される方程式系を解くことと同値であり,方程式系の解を得るには，適切な分岐被覆空間を考え,漸化式の中の局所的な留数の和をその上の大域的な線積分で表す必要がある。
私はこの分岐被覆空間を，第二構造接続と呼ばれる接続のモノドロミー被覆として導入することを提案し、モノドロミー被覆が特に古典的なリーマン面になるような半単純フロベニウス多様体を分類する技術を開発した。

研究成果の概要（英文）：The main focus of my research is on reconstructing certain type of Gromov－Witten invariants. The problem is equivalent to solving a system of equations defined by the so-called screening operators. My approach is based on the Eynard-Orantin recursion. The definition of the recursion involves a sum of residues of local meromorphic 1-forms. In order to obtain a solution to the screening equations we have to express the sum of local residues as a global contour integral and degenerate the branched covering. My proposal for branched covering is the monodromy covering space of the so-called second structure connection. I proved that this choice is correct for all simple singularities. In general, the monodromy covering space is not a classical Riemann surface, but some infinite sheet covering of P1. I developed a technique to classify semi-simple Frobenius manifolds for which the monodromy covering space is a classical Riemann surface.

研究分野：Representation theory

キーワード：period integrals　vertex algebras　Gromov-Witten invariants

## １．研究開始当初の背景

The Gromov-Witten (GW) invariants of a symplectic manifold X are certain intersection numbers on the moduli space of stable maps from Riemann surfaces to X. In case X is an algebraic variety they have an enumerative meaning: the number of curves satisfying various incidence and tangency constraints. In general, computing GW invariants is a very difficult problem. However, in the case when X has <u>semi-simple quantum co-homology</u> a great progress was made and a very strong connection with representation theory of infinite dimensional Lie algebras and integrable hierarchies was established. The GW invariants of X are organized in a generating function $D_X$, called the <u>total descendant potential</u> of X. It is a formal series, viewed as an element of a Fock space, whose coefficients are the GW invariants of X. It was conjectured by Givental [7] and proved by Teleman [13] that under the semi-simplicity assumption the higher genus invariants of X can be reconstructed from the genus-0 ones and the higher genus theory of the point. This result has two spectacular applications. The first one is to prove (see [8]) the so-called <u>Virasoro conjecture</u>. The latter is the main open problem in GW theory. It was proposed by Egouchi-Hori-Xiong [5] and it says that $D_X$ is a highest weight vector for a certain representation of the Virasoro algebra. The second application is to characterize $D_X$ as a solution to an <u>integrable hierarchy</u>. Motivated by GW theory, Dubrovin and Zhang have introduced a new class of integrable hierarchies [4]. In the settings of GW theory with semi-simple quantum co-homology the higher-genus reconstruction was used by Buryak-Posthuma-Shadrin to prove that the flows of the Dubrovin-Zhang's hierarchy are given by differential polynomials. Hence we have a very nice class of integrable systems that govern the intersection theory on the moduli spaces of stable maps. In many cases, due to a mirror symmetry phenomenon, one can express the GW invariants in terms of period integrals in the sense of K. Saito's theory of primitive forms [12]. One of my long-term goals is to obtain Hirota bi-linear equations for the tau-functions of the Dubrovin-Zhang's hierarchies. I managed to do this for simple singularities as well as for the projective line and some of its orbifold versions [9, 10]. In general, however the problem is quite challenging and it looks that new ideas are necessary. Recently in collaboration with B. Bakalov [1], I proved that in the settings of simple singularities $D_X$ is a highest-weight vector of a W-algebra. The latter is a certain Vertex Operator Algebra (VOA) extension of the Virasoro algebra defined via co-homological methods. It is a mysterious object that for some reason appears quite often when we have to deal with moduli problems in geometry. Using K. Saito's period integrals, we managed to construct a twisted representation of the lattice vertex algebra associated with the Milnor lattice. Moreover our representation has some global analytic properties, which in particular allowed us to prove the W-constraints for $D_X$.

## ２．研究の目的

he main purpose of this proposal was to investigate whether the local recursion can be extended to a global one. Furthermore, if the global recursion exists then we would like to relate it to the representation theory of W-algebras.

## ３．研究の方法

Let *f(x)* be a holomorphic function on 3 variables, s.t., the hyper-surface *f(x)=0* has an isolated singularity of type $X_N$, where X=ADE. We denote by *H* the middle homology of the Milnor fiber $f^{-1}(1)$, i.e., the hyper-surface in $\mathbf{C}^3$ defined by the equation $f(x)=1$. We are interested in the period integrals $y(t,\lambda)=\int \omega/dF_t$, where $F_t$ is a miniversal deformation of $f$ ($F_t$ at $t=0$ coincides with $f$) and $\omega$ is the standard volume form in $\mathbf{C}^3$. The integration is performed against cycles on the hyper-surface in $\mathbf{C}^3$ defined by the equation $F_t(x)=\lambda$. Let us fix a deformation parameter $t$ and a path from $(0,1)$ to $(t,\lambda)$ avoiding the discriminant locus (i.e. the points $(t,\lambda)$ where the Milnor fiber is singular). Using parallel transport along the path with respect to the Gauss-Manin connection, we may view $y(t,\lambda)$ as a linear function on $H$, i.e., an element in the dual space $H^*$. In other words, $y(t,\lambda)$ is a multi-valued function in $\lambda$ with values in $H^*$. It is not hard to see that the values of $y(t,\lambda)$ give rise to a Riemann surface $C_t \subset H^*$ which is also a branched covering of $\mathbf{C}$ branched at the critical values of $F_t$. Applying the Eynard-Orantin theory [6] with spectral curve $C_t$ we can define correlators and respectively a partition function.

On the other hand, it is known that $\omega$ is a primitive form in the sense of K. Saito, so we can equip the space of miniversal deformations with a semi-simple Frobenius structure and use Givental's higher-genus reconstruction to define respectively a total descendant and a total ancestor potentials $D_X$ and $A_{X,t}$ of the singularity X. Our expectation is that the partition function defined by the EO recursion coincides with $A_{X,t}$. In order to prove this we just have to identify the EO recursion with the local EO recursion that I proved recently [11]. Furthermore, the EO recursion is defined via summing up the residues of certain recursion kernel at all branching points. Following the approach of Bouchard-Eynard [2] we would like to construct a global EO recursion. The main difficulty is to find an appropriate recursion kernel, such that, we can replace the sum of the residues by a single contour integral that goes around all branching points. This way we can pass to the limit $t\to 0$, which should yield the W-constraints for the total descendant potential $D_X$ that we proved in [1]. This approach was applied successfully for $A_2$-singularity in [16] in order to prove a conjecture of Givental that the total ancestor potential $A_{X,t}$ depends analytically on the deformation parameter $t$. The generalization to an arbitrary singularity of type $A$ is straightforward. It remains only to modify the argument appropriately so that it works for singularities of type $D$ and $E$.

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## ４．研究成果

The main purpose of the proposal was achieved for the case of Frobenius manifolds corresponding to singularity of type A. Given a semi-simple Frobenius manifold we have proposed a spectral curve for the Eynard—Orantin recursion as a monodromy covering space. We proved that such a choice works in the case of all simple singularities. Finally, we have investigated the cases in which the monodromy covering space is a classical Riemann surface. The outcome is a certain type of Hurwitz Frobenius manifolds. We have developed a theory of primitive forms which one can use to study the relation between Frobenius manifolds and Eynard—Orantin recursion. In particular, we found several interesting applications to mirror symmetry.

There are several research groups abroad working on the Eynard—Orantin recursion and its application to geometry. The field is developing very quickly. In particular, there was a new development in 2017 due to Kontsevich and Soibelman who have interpreted the Eynard—Orantin recursion as an example of their general theory of deformation quantization. My research projects are very good test whether the new emerging ideas are interesting and worth further investigation. I think that the main challenge for the future is to generalize the recursion to infinite sheet coverings and use it to compute Gromov—Witten invariants of smooth projective varieties of dimension more than 1.

## ５．主な発表論文等
**（研究代表者、研究分担者及び連携研究者には下線）**

**〔雑誌論文〕（計 5 件）**

① Todor Milanov and Danilo Lewanski.
” *W-algebra constraints and topological recursion for $A_N$-singularity* “, Internat. J. Math. 27(2016), no.13, 1650110, 21 pp. Refereed,
DOI: 10.1142/S0129167X1650110X

② Todor Milanov, H.-H. Tseng, and Y. Shen.
“*Gromov-Witten theory of Fano orbifold curves, gamma integral structures and ADE-Toda hierarchies”,* Geome. Topol. 20(2016), no.4, p. 2135–2218. Refereed
DOI: 10.2140/gt.2016.20.2135

③ Todor Milanov and Yefeng Shen.
*“Global mirror symmetry for invertible simple elliptic singularities”.*
Annales de l'institut Fourier, 66 no. 1 (2016), p. 271-330. Refereed

④ Todor Milanov.
" *The Eynard-Orantin recursion for simple singularites* ". Communications in Number Theory and Physics, vol. 9(2015), no. 4, p. 707-739.　Refereed

⑤ T. Milanov and Y. Shen.
"The modular group for the total ancestor potential of Fermat simple elliptic singularities." Comm. In Number Theory and Phys., Vol. 8(2014), no. 2, p. 329-368 Refereed

〔学会発表〕（計 12 件）

① Todor Milanov

Title: Eynard--Orantin recursion for simple singularities of type A
Conference: Geometric and algebraic aspects of integrability
Place: Durham University, Durham, UK
Date: 2016.07.28 - 2016.08.04

② Todor Milanov

Title: K-theoretic Gromov--Witten invariants in genus 0 and integrable hierarchies
Conference: Categorical and Analytic invariants in Algebraic Geometry IV
Place: Kavli IPMU, Kashiwa, Chiba
Date: 2016.11.14 - 2016.11.18

③ Todor Milanov

Title: Hurwitz Frobenius manifolds and K. Saito's primitive forms
Conference: Interactions between topological recursion, modularity, quantum invariants, and low-dimensional topology
Place: Melbourne University, Creswick Campus, Victoria, Australia
Date: 2016.12.18 - 2016.12.23

④ Todor Milanov

Title: Vertex algebras in singularity and Gromov--Witten theory
Conference: Mirror Symmetry, Hodge Theory and Differential Equations
Place: Mathematisches Forschungsinstitut Oberwolfach Oberwolfach, Germany
Date: 2015.04.18 - 2015.04.26

⑤ Todor Milanov

Title: The Eynard--Orantin recursion for simple singularities
Conference: Geometric Invariants and Spectral Curves
Place: Lorentz Center, Leiden, Netherlands
Date: 2015.05.31 - 2015.06.07

⑥ Todor Milanov

Title: Vertex algebras and Gromov--Witten invariants
Conference: Categorical and Analytic invariants in Algebraic Geometry 1
Place: Steklov Institute, Moscow, Russia
Date: 2015.09.13 - 2015.09.19

⑦ Todor Milanov

Title: Mirror symmetry and the global ancestor potential
Conference: Primitive forms, mirror symmetry, and related topics 2014
Place: Kyoto University, Kyoto, Japan
Date: 2014.12.24

〔図書〕（計 0 件）

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〔その他〕
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