様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

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研究課題名（英文）Conjectures associated with Brascamp-Lieb type inequalities

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研究成果の概要（和文）：本研究の一つの重要な結果は非線形ブラスカンプ・リーブ(BL)予想の完全な証明に成功した事である。これは本研究の主な目的の一つで、一連の論文を通し達成する事ができた。最初に標準的なBL不等式の定数の局所有界性を示し、その結果を応用する事で任意に小さいソボレフの滑らかさをもつ関数に対する非線形BL予想を証明した。この安定性の結果の更なる応用として、多重線形フーリエ制限問題の理論を発展させた。次に、標準的なBL不等式の定数の連続性を証明し、それを用いる事によって非線形BL予想の完全な解決に成功した。更に、本研究では運動輸送方程式に関する評価を始めとする、様々な関連する理論を発展させる事に成功した。

研究成果の学術的意義や社会的意義
非線形ブラスカンプ・リーブ予想を証明した事により、抽象調和解析や偏微分方程式への応用を行った。例えば、リー群に関する局所的なヤングの畳み込み不等式に対する最良定数を求めた。更に、ザハロフ方程式系の理論における幾何的な不等式を示す事にも成功した。このように本研究の成果は、将来的にも多分野で応用される事が期待される。

調和解析および幾何解析の研究交流を更に推進するために、4日間の国際的な研究集会を東京ステーションカレッジ・埼玉大学サテライトキャンパスで開催した。

研究成果の概要（英文）：A key outcome of this research project was the successful complete verification of the nonlinear Brascamp-Lieb (BL) conjecture. This was one of the main goals of the initial research proposal and has been achieved through a series of papers. The first step was to establish local boundedness of the BL constant in the standard version of the BL inequality and use this to prove the nonlinear BL conjecture for input functions with arbitrarily small Sobolev regularity. As a further application of this stability result, we advanced the theory of the multilinear Fourier restriction problem. Next, we showed continuity of the BL constant and this result was used in our recent complete resolution of the nonlinear BL conjecture. Additionally, this research project has produced various new developments on related theory, including estimates for the kinetic transport equation.

研究分野： 調和解析

キーワード： Brascamp-Lieb不等式　安定性

１．研究開始当初の背景

(1) Hölder's inequality is a classical inequality in mathematics that provides control on the size of the multiplication of two functions in terms of Lebesgue space norms. Convolution is a fundamental concept in mathematics and is another type of product of two functions. The corresponding inequality which provides control on the size of the convolution of two functions in terms of Lebesgue space norms is called Young's convolution inequality and one may derive this inequality from Hölder's inequality. Another important consequence of Hölder's inequality is the Loomis-Whitney inequality. The Loomis-Whitney inequality was originally motivated by the famous isoperimetric problem from geometry concerned with maximising volumes of objects subject to a fixed surface area. In the mid 1970s, H. J. Brascamp and E. H. Lieb put forward a natural and common generalisation of each of the three aforementioned inequalities and established a number of important results concerning their inequality.

(2) An elegant and powerful theory of the Brascamp-Lieb inequality has emerged since its formulation with fundamental contributions in papers by Ball, Lieb, Barthe, Carlen-Lieb-Loss and Bennett-Carbery-Christ-Tao. Ball pioneered use of the Brascamp-Lieb inequality in convex geometry, a particularly well-known application being the use of the geometric form of the Brascamp-Lieb inequality to make dramatic progress on the problem of finding sharp bounds on the size of slices of cubes by lower-dimensional subspaces. Applications and viewpoints on the Brascamp-Lieb inequality may be found more widely in, for example, probability, stochastic processes, information theory and theoretical computer science. The Brascamp-Lieb inequality also plays a fundamental role in the multilinear Fourier restriction and Kakeya problems, as well as the closely related decoupling theory; the latter having seen staggering applications in recent years to problems in number theory (such as the Vinogradov Mean Value Theorem established by Bourgain-Demeter-Guth) and partial differential equations (such as the sharp Strichartz estimates for tori due to Bourgain-Demeter).

２．研究の目的

(1) The nonlinear version of the Brascamp-Lieb inequality, in particular, the nonlinear Loomis-Whitney inequality, emerged in work of Bennett-Carbery-Wright around 2005 in connection to multilinear Fourier restriction theory, and a few years later in work of Bejenaru-Herr-Tataru on the three-dimensional Zakharov system. A number of subsequent applications of more general nonlinear Brascamp-Lieb inequalities appeared in work of Bennett-Bez in connection to multilinear Fourier restriction theory, and several further papers on the Zakharov system and related partial differential equations.

(2) The initial and main purpose of the research project was to establish results on the stability of the standard form of the Brascamp-Lieb inequality. As well as advancing the theory of the standard Brascamp-Lieb inequality, it was anticipated that progress in this direction would lead to progress on the major goal of proving the so-called nonlinear Brascamp-Lieb conjecture. Certain special cases of the nonlinear Brascamp-Lieb conjecture were already known (such as the aforementioned nonlinear Loomis-Whitney inequality), but a fully general version was a major open problem in this field.

(3) In addition to the nonlinear Brascamp-Lieb conjecture, a further goal of this research project was to advance the theory of related multilinear inequalities, such as multilinear Fourier restriction estimates.

３．研究の方法

(1) In order to understand the stability of the constant in the standard Brascamp-Lieb inequality, two fundamental results concerning the Brascamp-Lieb constant due to Lieb and Bennett-Carbery-Christ-Tao were used. The work of Lieb establishes that the Brascamp-Lieb constant is exhausted by gaussian input functions and is a cornerstone in the theory. The latter result addresses a problem left open by Lieb's theorem, namely, to establish a characterisation of when the Brascamp-Lieb constant is finite.

(2) The main method underlying the approach to the nonlinear Brascamp-Lieb conjecture was induction-on-scales, first pioneered by Bourgain in harmonic analysis. A rough version of induction-on-scales yielded progress on the nonlinear Brascamp-Lieb conjecture where the input functions possess (arbitrarily small) Sobolev regularity.

(3) In the case where a suitable family of gaussian maximisers for the underlying standard Brascamp-Lieb inequality exist (such as the so-called subcritical case), a tight induction-on-scales argument yielded the nonlinear Brascamp-Lieb conjecture.

(4) In order to proceed to the fully general case and solve the nonlinear Brascamp-Lieb conjecture, it was necessary to establish a certain stronger (quantitative) form of Lieb ' s theorem in order to obtain a suitable family of gaussian near-maximisers. This argument used methods from optimisation theory.

４．研究成果

(1) The general case of the nonlinear Brascamp-Lieb conjecture was proved. In addition, applications of such nonlinear Brascamp-Lieb inequalities were made to problems in abstract harmonic analysis and partial differential equations. For example, the sharp constant for the local version of Young ' s convolution inequality on Lie groups was found. In addition, certain geometric inequalities arising in the theory of Zakharov systems were obtained. These inequalities provide bounds on iterated convolutions of Lebesgue space densities supported on submanifolds of euclidean space satisfying a transversality condition. These results are contained in joint work with Jonathan Bennett, Stefan Buschenhenke, Michael Cowling and Taryn Flock (arXiv:1811.11052), currently under review.

(2) Results regarding the stability of the constant in the standard Brascamp-Lieb inequality have been obtained, initially a local boundedness result and then subsequently strengthened to a continuity result. Such stability results have played a role in the development of decoupling theory in recent years and hence the subsequent applications to problems in number theory.

(3) Results have been obtained for the refinement of the standard Brascamp-Lieb inequality where the input functions belong to Lorentz spaces. By obtaining certain new necessary conditions, we established the sharpness of a result of Christ concerning the range of allowable exponents for the Lorentz spaces in the so-called subcritical case.

(4) This research project has also contributed to the theory certain partial differential equations, including the kinetic transport equation and the free Schrödinger equation. For example, smoothing estimates for the velocity average of the solution to the kinetic transport equation have been significantly advanced to a mixed-norm setting. This has been achieved through the use a variety of techniques from harmonic analysis, including Fourier restriction theory, decoupling estimates for the cone, and bilinear interpolation theory.

(5) To promote greater interaction and collaborative research between mathematicians working in harmonic analysis and geometric analysis, within Japan and internationally, a 4-day international workshop was organized.

2016年11月28日～2016年12月1日, Interactions Between Harmonic and Geometric Analysis, 国際研究集会, 埼玉大学・東京ステーションカレッジ・埼玉大学サテライトキャンパス

５．主な発表論文等

〔雑誌論文〕（計８件）

① Jonathan Bennett, Neal Bez, Generating monotone quantities for the heat equation, Journal fur die reine und angewandte Mathematik, 印刷中. (査読有)

② Neal Bez, Chris Jeavons, Tohru Ozawa, Mitsuru Sugimoto, Stability of trace theorems on the sphere, Journal of Geometric Analysis, Vol. 28 (2018), pp. 1456-1476. (査読有)
DOI:10.1007/s12220-017-9870-8

③ Jonathan Bennett, Neal Bez, Susana Gutierrez, Sanghyuk Lee, Estimates for the kinetic transport equation in hyperbolic Sobolev spaces, Journal de Mathematiques Pures et

Appliquees, Vol. 114 (2018), pp. 1-28. (査読有)
DOI:10.1016/j.matpur.2018.03.007

④ Neal Bez, Jayson Cunanan, Sanghyuk Lee, Smoothing estimates for the kinetic transport equation at the critical regularity, SIAM Journal on Mathematical Analysis, Vol. 50 (2018), pp. 2280-2294. (査読有)
DOI:10.1137/17M1148852

⑤ Jonathan Bennett, Neal Bez, Taryn Flock, Susana Gutierrez, Marina Iliopoulou, A sharp k-plane Strichartz inequality for the Schrödinger equation, Transactions of the American Mathematical Society, Vol. 370 (2018), pp. 5617-5633. (査読有)
DOI:10.1090/tran/7309

⑥ Jonathan Bennett, Neal Bez, Taryn Flock, Sanghyuk Lee, Stability of the Brascamp-Lieb constant and applications, American Journal of Mathematics, Vol. 140 (2018), pp. 543-569. (査読有)
DOI:10.1353/ajm.2018.0013

⑦ Jonathan Bennett, Neal Bez, Michael Cowling, Taryn Flock, Behaviour of the Brascamp-Lieb constant, Bulletin of the London Mathematical Society, Vol. 49 (2017), pp. 512-518. (査読有)
DOI:10.1112/blms.12049

⑧ Neal Bez, Sanghyuk Lee, Shohei Nakamura, Yoshihiro Sawano, Sharpness of the Brascamp-Lieb inequality in Lorentz spaces, Electronic Research Announcements in Mathematical Sciences, Vol. 24 (2017), pp. 53-63. (査読有)
DOI:10.3934/era.2017.24.006

〔学会発表〕(計 19 件)

① Neal Bez, Smoothing estimates for velocity averages, 九州大学九州関数方程式セミナー, 2018 年

② Neal Bez, Geometric estimates arising in the analysis of Zakharov systems, 特別講演, 日本数学会 2018 年度秋季総合分科会, 2018 年

③ Neal Bez, Mixed-norm estimates for multipliers associated with some hypersurfaces, Harmonic Analysis and PDE Workshop, University of Birmingham, 2018 年

④ Neal Bez, Strichartz estimates for orthonormal systems of initial data, International Workshop on Fundamental Problems in Mathematical and Theoretical Physics, 早稲田大学, 2018 年

⑤ Neal Bez, On the kinetic transport equation, 調和解析と非線形微分方程式, 京都大学数理解析研究所, 2018 年

⑥ Neal Bez, Smoothing estimates for the kinetic transport equation at the critical regularity,保存則をもつ偏微分方程式の解の正則性, 特異性および漸近挙動の研究, 京都大学数理解析研究所, 2018 年

⑦ Neal Bez, Multilinear inequalities in harmonic and geometric analysis, Topics in Harmonic Analysis: Intensive Lecture Series, Seoul National University, 2017 年

⑧ Neal Bez, Strichartz estimates for orthonormal systems of initial data, Nonlinear Partial Differential Equations for Future Applications: Hyperbolic and Dispersive

PDE，東北大学，2017年

⑨ Neal Bez, On the kinetic transport equation, Harmonic Analysis and its Interactions: in Honour of Tony Carbery, International Centre for Mathematical Sciences, 2017年

⑩ Neal Bez, Stability of the Brascamp-Lieb constant and applications, International Conference for the 70th Anniversary of the Korean Mathematical Society, Seoul National University, 2016年

〔図書〕（計0件）

〔産業財産権〕
○出願状況（計0件）

○取得状況（計0件）

〔その他〕
ホームページ等
http://www.rimath.saitama-u.ac.jp/lab.jp/nealbez/Main.html

６．研究組織

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