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研究成果の概要（和文）：Positive Representationsに関する研究を行った。主にPositive Representationsのクラスター構造を発見して、普遍R行列の分解を作った。それを使って、A型の場合、最も重要な予想二つがやっと証明した。特に、Borel部分のテンソル積分解を明示的に書けて、さらに、シレダ・シャピロによってPositive Representationsのテンソル積分解も同じようなアイデアで証明した。彼らとの共同研究でPeter-Weyl定理も同様に証明した。
また、ペナー・ゼットリンの共同研究によって、N=2 タイヒミュラー超空間を作って、奇次元など色んな性質を研究した。

研究成果の概要（英文）：We discovered a cluster realization of the positive representations of split real quantum groups and factorization of universal R operator. With this new result, we found explicitly the tensor product decomposition of positive representations restricted to the Borel part. The same technique is further developed by Schrader-Shapiro to prove that the positive representations in type An is closed under taking tensor product by identifying the Casimirs with the open Coxeter-Toda Hamiltonians, proving part of the main conjecture started for this project. In an upcoming work, we also show that the Peter-Weyl theorem holds using the same decomposition.

With R. Penner and A. Zeitlin, we also constructed the N=2 super decorated Teichmuller theory, and described the dimension reduction of the odd coordinates of the super Teichmuller space due to the constraint arising from Ramond punctures on the surface, so that the result is compatible with that of the moduli spaces of super Riemann surfaces.

研究分野：Representation Theory

キーワード：Positive representation　quantum groups　modular double　Teichmuller theory　cluster algebra　integrable systems

１．研究開始当初の背景

Finite dimensional representation theory of quantum groups introduced by Drinfeld and Jimbo in 1985, originally motivated for the search of the solutions to the Yang-Baxter equation, has led to many important applications in mathematics and physics over 30 years. In particular the braided tensor category structure gives rise to various 3-manifold and knot invariants, including the famous Reshetikhin-Turaev's TQFT. The finite dimensional representation theory corresponds to one of the real form called the *compact form* in Lie theory. On the other hand, the representation theory of another important real form, the *split real form*, and its corresponding quantization is much less understood.

The notion of ***positive representations*** was initiated in a joint work with I. Frenkel (2012) as a new research program devoted to the study of representation theory of split real quantum groups. In the simplest case of $U_q(sl(2,\square))$ studied by Teschner *et al.*, it demonstrates a lot of similarities with the finite dimensional representation theory of the compact case. Most importantly, it shows a braided tensor category structure in a continuous sense, and I have established previously (Ip, 2012) a Peter-Weyl type theorem relating positive quantized functions $L^2(SL_q^+(2,\square))$ with the regular representations of $U_q(\mathfrak{sl}(2,\square))$. However, at the same time it also provides many new phenomena such as Faddeev's modular double, a simple analytic relation giving the Langland's duality, positivity properties and their relations to cluster algebra, and the use of quantum dilogarithm function that is not available in the compact case.

We continue to study this special class of representations, with the main focus on the braided tensor category structure of the positive representations and its applications, further developing the results from the previous project (KAKENHI No.26800004). In this project we focus on the newly discovered cluster algebraic realization of the positive representations and factorization of universal R operator. This approach is closely related to the quantum Teichmüller theory, where mutations of a triangulations on a marked surface is realized by conjugation by the quantum dilogarithm functions, thus providing us a unitary equivalence of the corresponding representations. This allows us to identify the mutation sequence needed to decompose the tensor product of positive representations restricted to the Borel part, and further decompose the representation itself in type A_n by identifying the Casimir operators with Hamiltonians of certain integrable systems.

Finally, we also continue our study of the theory in the super case by constructing the $N=2$ decorated super-Teichmüller space and study its various properties, including the odd dimension reduction around Ramond punctures, which were partly done in the previous project.

２．研究の目的

In this project, we develop further the theory of positive representations of split real quantum groups. The aim of the project is to try to establish the tensor product decomposition structure of positive representations, and develop its applications to other field of mathematics. More precisely, we

a) Construct a cluster algebraic realization of the positive representation of split real quantum groups and factorization of R matrix, following the recent idea by Schrader-Shapiro in the case of type A_n. This allow us to apply cluster algebraic techniques to study mutations of the representations, the tensor product of the representations as a concatenation of diagrams, and the Casimirs operator as the monodromy around the punctures of the associated marked surface.

b) Develop the properties for the theory of $N=2$ decorated super-Teichmüller space, generalizing the $N=1$ construction by Penner-Zeitlin. By appropriately quantizing this space, this serves as a starting point to understand the notion of super-cluster algebra, and application of the above cluster realization technique to develop a new theory of split real quantum groups in the super case.

３．研究の方法

I have been travelling to conferences and workshops in Mathematics (Representation Theory) and Physics (Integrable Systems) to give seminars and promote the idea of positive representations of split real quantum groups and its applications. In particular, the newly developed idea of the cluster realization of split real quantum groups was presented in the Mathematics Society of Japan Spring Meeting 2017 as a special invited lecture (**特別講演**). Many of the research ideas in the latter half of the project was developed during a visit to

Toronto and discussion with G. Schrader, A. Shapiro and I. Le, who were also invited to Kyoto University for further discussion concerning the positive Peter-Weyl theorem, and among other projects the decomposition of Casimirs in other types, including S_3 action of the cluster structure associated to triangulations, and Knudson-Tao's hive condition in other types which will remain to be done in a future project.

In summary, the proposed method for aims of the project outlined in the previous section are as followed:

a) Following the idea of Schrader-Shapiro in type A_n, we develop the cluster realization of split real quantum groups. By forgetting the real structure, one can find using the explicit formula for the positive representations, that there exists a polynomial embedding of quantum group $U_q(\mathfrak{g})$ into the quantum torus algebra associated to a punctured disk with two marked points. The change of longest word can now be easily identified with quiver mutations. Furthermore, the universal R operators constructed previously, can be decomposed into products of quantum dilogarithm thanks to the new realization of the generators of $U_q(\mathfrak{g})$, and these products can be identified with the quiver mutations realized as a half-Dehn twist, corresponding to 4 flips of triangulations of the marked surface.

   With this new realization, various properties of the positive representations can be studied using cluster algebra technique and combinatorically using quiver mutations, which are represented using conjugations of the quantum dilogarithm functions, and thus giving us various unitary transformations between different forms of the representations. Furthermore, the two main problems of this project, specifically, the tensor product decomposition as well as the Peter-Weyl theorem, can now be viewed as merely combinatorial problems of finding nice sequences of quiver mutations to bring certain quiver diagrams associated to triangulations of punctured surface with marked points to a desired form for the Casimir to be decomposed. This will be explained in detail in the next section.

b) Together with R. Penner and A. Zeitlin, we generalize the construction of the $N=1$ super-Teichmüller space of flat $OSp(1|2)$ connections on a punctured surface, we extend to the case $N=2$ of flat $OSp(2|2)$ connections. The construction simplify the previous treatment in the case of $N=1$, and also serves as a nontrivial starting point of constructing higher super-Teichmüller space with super-group. The main ingredient in the construction is the derivation of the analogue of Ptolemy transformations for the new coordinates on the super-space, which motivates us to utilize this construction to develop further the correct notion of super-cluster algebra on punctured surface, generalizing the cluster mutation formula of the usual Ptolemy relations in the classical setting.

４．研究成果

a) The main result is the cluster realization of split real quantum groups, which serves as the technical backbone of several important goals of this project, including the tensor product decomposition and the Peter-Weyl theorem. For each simple Lie type $\mathfrak{g}$, given a triangulation on a punctured surface with marked points, one can associate to each triangle a quiver diagram Q. This coincides with the cluster X-variety structure of a framed local G-system on the surface. We showed that there exists an embedding of the quantum group $U_q(\mathfrak{g})$ into the quantum torus algebra associated to the quiver corresponding to the triangulations of the once-punctured disk with 2 marked points, such that the generators of $U_q(\mathfrak{g})$ can be represented by telescoping sums of the cluster variables along oriented paths on the quiver. By giving a polarization of the cluster variables, we recover the positive representations of the split real quantum groups.

   The tensor product of positive representations can then be realized as amalgamating the quivers to form the one corresponding to a 2-punctured disk with 2 marked points, and naturally the action of the universal R matrix, which permutes the tensor factors, can be realized as quiver mutations corresponding to 4 flips of triangles producing the so-called *half-Dehn twist*. The explicit mutation sequence to flip a triangle is new and is the most important ingredient to proving several results.

   We showed that the embedding to the

quantum torus algebra associated to each triangle corresponds to the *Heisenberg double* of the Borel part of $U_q(\mathfrak{g})$, and hence show that the tensor product of positive representations restricted to the Borel part can be decomposed by flipping the triangulation of the twice-punctured disk producing self-folded triangles. This gives explicit mutation sequences for the tensor product decompositions, generalizing the results in the previous project (KAKENHI No.26800004) where instead only abstract construction from C^*-algebra is utilized.

Furthermore, one observed that the monodromy around punctures encoded the central characters of the positive representations. By going further from the Borel decomposition, one can flip the triangulations so that the Casimir operators lie on an annulus surrounding the 2 punctures. This identifies the Casimir as the trace of a monodromy matrix lying in the quantum torus algebra generated by the cluster variables associated to the double big Bruhat cell. In type A_n, Schrader-Shapiro observed that this can be further reduced to the double Coxeter cell, thus identifying the Casimirs as Hamiltonians of the open Coxeter-Toda systems. Since these can be decomposed by the so-called *q-Whittaker functions*, the tensor product of positive representations can be decomposed, proving the long standing conjecture in type A_n which is one of the main goals of this project.

Finally, the same technique of the mutations can also be applied instead to a cylinder with 4 marked points, where one can identify it with the Hilbert space $L^2(SL_q^+(n,\mathbb{R}))$, and the surprising result is that this corresponds to the statement of the Peter-Weyl theorem in the split real case, where $L^2(SL_q^+(n,\mathbb{R}))$ is decomposed into a direct integral of the left and right regular representation of $U_q(\mathfrak{sl}(n,\mathbb{R}))$. More precisely, one shows that the Casimir operators for both the left and right regular representation is given by the trace of the monodromy matrix around the annulus, which shares exactly the same formula as in the previous case. Therefore the cluster algebraic technique together with the decomposition of the Casimir operators, solve the 2 main goals of this project at the same time.

b) Following the idea by Penner-Zeitlin in the case of $N=1$, we constructed the $N=2$ decorated super-Teichmüller space as follows. We defined the lightcone to be the $OSp(2|2)$-orbit of the highest weight vector of the adjoint action on a superspace $\mathbb{R}^{2,2|4}$, and studied the orbits of quadruples of points on the lightcone, which unlike the $N=1$ case, has an extra even coordinates called the ratio of two triangles. With this description, we defined the decorated super-Teichmüller space to be the moduli space of lifts from the universal cover of the punctured surface to the lightcone, and describe the coordinate system of its components explicitly, namely they are parametrized by 2 even coordinates for each edge, and 2 odd coordinates for each triangle for the triangulation of the punctured surface, such that it is π_1-equivariant for the super-Fushian representation. The extra coordinate is an $\mathbb{R}_+$-connection on the fat graphs dual to the triangulations, which also encode the 2 spin structures for the Teichmüller space. Finally the main result is the explicit description of the Ptolemy relations among the 8 coordinates with respect to a flip of triangulations, generalizing the formula for the $N=1$ case. In particular, we expect this new formula together with the previous one will give us a correct way to define what should be a super-cluster algebra structure. We remark that attempts have been done by several people previously, however, with the mutations of the odd variables somehow ignored, which we believe should not be the case.

The $N=2$ decorated super-Teichmüller space we defined above turn out has extra odd dimensions compared with the usual moduli space of super-Riemannian manifold. To compare the two space, in the second joint publication, we study the conditions for the dimension reduction, and showed that a natural subspace is obtained by requiring that the $OSp(2|2)$-monodromy around punctures should be conjugated to the standard parabolic element of $SL(2,\mathbb{R})$ inside $OSp(2|2)$, thus reducing the overall odd dimension by the number of Ramond punctures, giving the correct dimension associated to the standard moduli space of supermanifold.

５．主な発表論文等
（研究代表者、研究分担者及び連携研究者には下線）

〔雑誌論文〕（計　5　件）

1) I. Ip,
On tensor products of positive representations of split real quantum Borel subalgebra $U_{q\tilde{q}}(\mathfrak{b}_{\square})$,
Trans. A.M.S., **370** (6), (2018), 4177-4200 (refereed)
http://dx.doi.org/10.1090/tran/7110

2) I. Ip,
Cluster Realization of positive representations of split real quantum Borel subalgebra,
Theo. Math. Phys. (to appear) (2018) (refereed)

3) I. Ip,
Cluster Realization of $U_q(\mathfrak{g})$ *and Factorization of Universal R Matrix,*
arXiv:1612.05641 (preprint)

4) I. Ip, R. Penner, A. Zeitlin,
N=2 Super Teichmüller Theory,
arXiv:1605.08094 (preprint)

5) I. Ip, R. Penner, A. Zeitlin,
On Ramond Decorations,
arXiv:1709.06207 (preprint)

〔学会発表〕（計 16 件）

1) I. Ip, *"Positive Casimir and Central Characters of Split Real Quantum Groups",*
Mathematical Physics and Geometric Analysis Seminar,
KIAS, Korea (2016 May)

2) I. Ip, *"Split Real Quantum Groups",*
The 24th International Conference on Integrable Systems and Quantum Symmetries,
Prague, Czech Republic (2016 June)

3) I. Ip, *"Positive representations of split real quantum groups",*
Mathematics Society of Japan Autumn Meeting 2016,
Kansai University (2016 Sept)

4) I. Ip, *"Positive representations: a bridge between Drinfeld-Jimbo quantum group and C*-algebra",*
Kyoto Operator Algebra Seminar, Kyoto University (2016 Nov)

5) I. Ip, *"Positive representations of split real quantum groups",*
Mathematics Colloqium,
HKUST (2016 Dec)

6) I. Ip, *"Cluster realization of* $U_q(\mathfrak{g})$ *and factorization of universal R matrix",*
Representation Theory Seminar,
RIMS (2017 Jan)

7) I. Ip, *"Cluster realization of* $U_q(\mathfrak{g})$ *and factorization of universal R matrix",*
The 2nd KTGU Mathematics Workshop for Young Researchers,
Kyoto University (2017 Feb)

8) I. Ip, *"Positive representations and cluster realization of quantum groups",*
Mathematics Society of Japan Spring Meeting 2017 (**特別講演**),
Tokyo Metropolitan University (2017 Mar)

9) I. Ip, *"Positive representations of split real quantum groups",*
Geometric Representation Theory Seminar,
University of Toronto, Canada (2017 Apr)

10) I. Ip, *"Positive representations of split real quantum groups",*
Scientific Seminar,
Perimeter Institute, Canada (2017 May)

11) I. Ip, *"Positive representations and cluster realization of quantum groups",*
Workshop on Classical and Quantum Integrable Systems,
Dubna, Russia (2017 Jul)

12) I. Ip, *"Generalized Teichmüller Spaces, Spin Structures, and Ptolemy Transformations",*
Workshop on Representation Theory of Lie Superalgebras and Related Topics,
Taipei, Taiwan (2017 Jul)

13) I. Ip, *"On tensor product decomposition of positive representations",*
Mathematics Society of Japan Autumn Meeting 2017,
Yamagata University (2017 Sept)

14) I. Ip, *"Cluster realization and tensor product decomposition of positive representation",*
Infinite Analysis 17,
Osaka City University (2017 Nov)

15) I. Ip, *"Cluster realization and tensor product decomposition of positive representations",*
International Symposium RIKKYO MathPhys 2018,
Rikkyo University (2018 Mar)

16) I. Ip, *"Positive Peter-Weyl Theorem",*
Representation Theory Seminar,
RIMS (2018 May)

〔図書〕（計　0　件）

〔産業財産権〕

○出願状況（計　0　件）

名称：
発明者：
権利者：
種類：
番号：
出願年月日：
国内外の別：

○取得状況（計　0　件）

名称：
発明者：
権利者：
種類：
番号：
取得年月日：
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〔その他〕
ホームページ等

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