様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

科学研究費助成事業　　研究成果報告書

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研究課題名（和文）Riemann-Hilbert problem for Gromov-Witten invariants

研究課題名（英文）Riemann-Hilbert problem for Gromov-Witten invariants

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研究成果の概要（和文）：本研究の背景は半単純フロベニウス多様体の理論である。量子コホモロジーと特異点の理論に発生したフロベニウス多様体は代表的な例である。任意の半単純フロベニウス多様体上の周期ベクトルの集合を構成し、頂点作用素という大変重要な微分作用素を導入した。頂点作用素の積の特異点の分析は本研究の最も大事な問題である。二つの特異点の近傍を任意の曲線でつなげると作用素の積はどんな風に変化するかと言う問題は非常に難しいため、解答は本研究の最も大事な結果である。単純特異点の場合、周期ベクトルの数値がK-理論的に記述できることは、本研究の二つ目の結果である。さらに、この二つの結果を用いて、様々な可積分系を構成した。

研究成果の学術的意義や社会的意義
The project gives new methods to construct differential equations with possible applications to physics, engineering, and cosmology. Highly specialized results in complex geometry are becomming more accesible to young researchers.

研究成果の概要（英文）：The results of this project are in the settings of the theory of semi-simple Frobenius manifolds. The main examples of such manifolds come from quantum cohomology and singularity theory. Motivated by Kyoji Saito's theory of primitive forms, we have introduced the notion of period vectors for any semi-simple Frobenius manifold. Using the period vectors and following ideas of Givental and Milanov we introduce vertex operators. The main result of this proposal is a connection formula for the operator product expansion (OPE) of the vertex operators, i.e., we found a general rule that allows us to analytically continue the OPE from one singularity to another one. The second main achievement of this proposal is a K-theoretic interpretation of the period integrals for simple singularities corresponding to vanishing cycles. Both results were applied to the problem of constructing integrable hierarchies of differential equations in the form of Hirota quadratic equations.

研究分野：complex geometry and differential equations

キーワード：period integrals　quantum cohomology　vertex operators　mirror symmetry

１．研究開始当初の背景

Frobenius manifolds were introduced by Dubrovin to give a geometric interpretation of the properties of quantum cohomology, that is, genus-0 Gromov-Witten invariants. Soon after Dubrovin's work it was realized that the notion of a Frobenius manifold was introduced in singularity theory by Kyoji Saito under the name flat structure. The abstract theory of Frobenius manifolds is developed mostly for the purposes of quantum cohomology and singularity theory. It was conjectured by Givental and proved by Teleman that if the quantum cohomology is semi-simple, then the higher-genus Gromov-Witten invariants are uniquely reconstructed by the genus-0 ones, that is, by the Frobenius structure underlying quantum cohomology. Givental's reconstruction is formulated in terms of a certain formal generating function known as the total descendent potential. The total descendent potential of a semi-simple Frobenius manifold is expected to be a tau-function of an integrable hierarchy. The latter was introduced by Dubrovin and Zhang in the form of bi-Hamiltonian equations. In fact, the construction of Dubrovin and Zhang is in some sense tautological. The Hamiltonians and the Poisson brackets of their hierarchy are defined in terms of Gromov-Witten invariants. In particular, finding an alternative, more explicit description of the hierarchy was the main goal of this proposal. Prior to the proposal, I worked out many examples both in Gromov-Witten theory and singularity theory. In collaboration with Bojko Bakalov, we have discovered an interesting relation between the semi-simple Frobenius manifolds for simple singularities and the representation theory of lattice vertex algebras. There was a technical difficulty in extending our results with Bakalov in general to arbitrary semi-simple Frobenius manifolds. Another relevant result, due to Hiroshi Iritani, was a K-theoretic description of the so-called integral structure in quantum cohomology.

２．研究の目的

The main goal of the proposal was to use Iritani's integral structure and the methods from the representation theory of vertex algebras to construct integrable hierarchies in the form of Hirota quadratic equations and to prove that the total descendent potential of Givental is a solution. Iritani's integral structure contains a certain discrete subset of vectors which have properties similar to root systems. This generalized root system contains information about the monodromy data of the Frobenius manifold. By expressing the Gromov-Witten invariants in terms of the monodromy data, we will obtain a solution of the Riemann-Hilbert problem involved in the theory of semi-simple Frobenius manifolds. In particular, that was the main reason for the title of the proposal.

３．研究の方法

The main idea was to investigate the properties of a certain set of vertex operators introduced first by Givental in the settings of singularity theory. The main ingredient in the construction are the period integrals of Kyoji Saito. In the abstract settings of Frobenius manifolds, Saito's period integrals are solutions to a Fuchsian system of differential equations known as the 2^{nd} structure connection. They are also compatible with the operation of stabilization. Using these two properties we were able to find an axiomatic definition of the notion of a period vector for any semi-simple Frobenius manifold. Another important notion in our method is the set of reflection vectors of a Frobenius manifold. In the set of all solutions to the second structure connection we have to single out those that correspond to period integrals of vanishing cycles. This is done naturally by making use of the monodromy properties of the second structure connection. The main fact is that the monodromy group is a reflection group, so we can introduce the reflection vectors by fixing their Laurent series expansions at the singularities of the second structure connection. The most challenging problem in the proposal is to understand the properties of the reflection vectors. In all examples in which the methods of the proposal can be applied successfully, the set of reflection vectors is either a classical root system or an affine root system. In this case, our vertex operators provide a representation of the corresponding Lie algebra. The construction of the corresponding Hirota quadratic equations can be compared with the construction of the so-called Kac-Wakimoto hierarchies.

４．研究成果

A semi-simple Frobenius manifold can be viewed as an isomonodromic deformation of a Fuchsian connection on $\mathbf{P}^1$ with singularities at infinity and at finitely many finite points in $\mathbf{P}^1$. Givental's higher genus reconstruction involves two operator series S and R. The former one determines a basis of solutions at infinity while the latter singles out a solution near each finite singular point corresponding to a reflection vector. The first problem in our project is to prove that the vertex operators corresponding to reflection vectors are pairwise local in the sense of the theory of vertex algebras, or equivalently in the sense of Wightman's axioms in quantum field theory. This is actually not so hard to do because it is sufficient to be done in a neighborhood of one singular point. We choose the point at infinity and the proof of the locality becomes a local computation. The next problem is to analyze the product of two vertex operators. A standard computation shows that the product consists of a vertex operator and a scalar function which we call the phase factor. The latter is a priori an infinite sum of quadratic expressions involving the period vectors. We proved that this infinite sum is convergent and that it can be analytically extended along any path in the Frobenius manifold avoiding the singularities. In other words, the phase factor is a multivalued analytic function. The first major result of the proposal is a connection formula for the phase factors. Namely, by using the R-matrix we can construct vertex operators corresponding to the reflection vector associated with each finite singularity and consider the corresponding phase factor. Similarly, near infinity we can use the S-matrix to construct a corresponding vertex operators and phase factors. Suppose now that we fix a path between inifinity and one of the singular points. Then we found a precise formula expressing the analytic continuation of the phase factor at the finite singular point to the phase factor at infinity. Our proof depends on the so-called Painleve property for semi-simple Frobenius manifolds. This is a rather deep result that follows from the work on Malgrange and the available proof in the literature requires quite spelized knowledge of functional analysis. On the other hand, there is a sketch of proof by Bolibruch which is much easier to follow. We made a small pedagogical contribution to the subject by filling in the details of Bolibruch's proposal. Establishing the connection formula is a mile stone for me. In all my previous works, whenever I had to prove that the total descendent potential is a tau-function for some integrable hierarchy, I had to work hard to obtain explicit formulas for the phase factors. The connection formula removes the need for explicit formulas, so now I have a general method for proving that the total descendent potential is a solution to a given system of Hirota quadratic equations (HQEs). The next step in my project is to understand the origin of the HQEs. The set of reflection vectors generate a lattice and by recalling the main result from my collaboration with Bakalov, we have that the vertex operators induce a representation of the lattice vertex algebra on the Fock space containing the total descendent potential. The HQEs should come from a vector in the lattice vertex algebra satisfying the so-called screening equations. Unfortunately, I still don't have a tool to construct solutions to the screening equations. Some new ideas are necessary here. Instead, I focused on understanding the complexity of the set of reflection vectors. More precisely, I would like to classify the set of reflection vectors in a way similar to the classification of classical root systems. In the settings of quantum cohomology, by reformulating Dubrovin's conjecture, we have a very precise conjecture for the set of reflection vectors in terms of the K-theoretic classes of exceptional objects in the derived category. The classification could be done according to the dimension of the target manifold. In dimension 1, everything is clear. All orbifolds with semi-simple quantum cohomologies must be orbifold projective lines. The exceptional objects are well understood in this case and our reflection vectors, at least conjecturally, form an extension of the real roots of a generalized Kac-Moody Lie algebra. I think that in this case the main goal of the proposal is achievable but yet not so easy. The representation theory of generalized Kac-Moody Lie algebras is a rather challenging topic. In dimension 2, the first target to investigate is $\mathbf{P}^2$. I was able to find explicit formulas for the period map of small quantum cohomology in terms of Eisenstein series and to find a perturbative algorithm to reconstruct the full period map. The set of reflection vectors looks very different from the root systems in the theory of Kac-Moody Lie algebras. In particular, the idea to choose a chamber in which the reflection hyperplanes have only locally finite intersections does not work. I think that finding a Lie theoretic interpretation of the reflection vectors of $\mathbf{P}^2$ is a very interesting problem. In collaboration with my student, Xiaokun Xia we made an interesting progress in proving that Dubrovin's conjecture for the reflection vectors is compatible with the blowup of finitely many points. In particular, my work for the projective plane could be generalized to the Del Pezzo surfaces. It will be interesting to work out the monodromy groups and to obtain a characterization of the corresponding reflection vectors. Finally, I was able to make an interesting progress in the settings of singularity theory too. In the case of quantum cohomology, we have a very powerful conjecture about the reflection vectors. In collaboration with my other student Chenghan Zha, I was able to find a similar description for the set of vanishing cycles for simple singularities. Namely, we can identify the Milnor lattice with a relative topological K-ring of a Berglund-Hubsch dual singularity. Our description makes sense also for all singularities corresponding to invertible polynomials. The topological K-ring could be viewed also as the K-ring of an appropriate category of equivariant graded matrix factorizations. At least for invertible polynomials, we can state a conjecture about the reflection vectors parallel to Dubrovin's conjecture but in which the derived category of complexes of coherent sheaves should be replaced with the derived category of matrix factorizations.

Unfortunately, I did not have time to pursue another promising idea based on the topological recursion of Eynard and Orantin. I was able to make a partial progress in the case of $\mathbf{P}^2$. Namely, I was able to construct a monodromy covering space for the spectral curve of the local recursion. Recall that the recursion kernel of the local recursion is defined in general but it is a multivalued analytic function. By constructing a monodormy covering space, we can lift the recursion kernel to obtain a single value analytic function. I was able to find explicit formulas for the pullback. However, the formulas look rather complicated and it is not easy to find a global recursion.

In the near future, I would like to continue my work on the problems of this proposal. Especially, the case of orbifold projective lines looks quite attractive. We can divide the orbifold projective lines according to the sign of their orbifold Euler characteristics. If the orbifold Euler characteristics is positive, then everything is done, except for 3 cases in which the set of reflection vectors form an affine root system of type E. The problem is to construct an extension of the corresponding Kac-Wakimoto hierarchy. I did not write a paper yet but recently in collaboration with Bakalov, we were able to construct the extension. If the orbifold Euler characteristics is 0, then we have elliptic root systems. I expect that we have to construct a double extension of the corresponding Kac—Wakimoto hierarchy which should not be that difficult. The true challenge will be when the orbifold Euler characteristics is negative. We have to work with generalized Kac—Moody Lie algebras. Nevertheless, there are new ideas coming from physics which I am currently investigating.

５．主な発表論文等

〔雑誌論文〕　計13件（うち査読付論文　13件／うち国際共著　8件／うちオープンアクセス　1件）

１．著者名 Todor Milanov and Alexis Roquefeuil	４．巻 409 B
２．論文標題 Confluence in quantum K-theory of weak Fano manifolds and q-oscillatory integrals for toric manifolds	５．発行年 2022年
３．雑誌名 Advances in Mathematics	６．最初と最後の頁 1-57
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.aim.2022.108682	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Jipeng Cheng, Todor Milanov	４．巻 388
２．論文標題 Hirota quadratic equations for the Gromov--Witten invariants of P_{n-2,2,2}^1	５．発行年 2021年
３．雑誌名 Advances in Mathematics	６．最初と最後の頁 -
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.aim.2021.107860	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Alexander Alexandrov, Todor Milanov	４．巻 111
２．論文標題 Matrix model for the total descendent potential of a simple singularity of type D	５．発行年 2021年
３．雑誌名 Letters in Mathematical Physics	６．最初と最後の頁 -
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s11005-021-01431-z	査読の有無 有
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１．著者名 Todor Milanov and Chenghan Zha	４．巻 16
２．論文標題 Integral structure for simple singularities	５．発行年 2020年
３．雑誌名 SIGMA	６．最初と最後の頁 28p
掲載論文のDOI（デジタルオブジェクト識別子） 10.3842/SIGMA.2020.081	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

１．著者名 Todor Milanov	４．巻 763
２．論文標題 Singularity theory and mirror symmetry	５．発行年 2021年
３．雑誌名 Contemporary Mathematics	６．最初と最後の頁 131-153
掲載論文のDOI（デジタルオブジェクト識別子） 10.1090/conm/763	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Hiroshi Iritani, Todor Milanov, Yongbin Ruan, and Yefeng Shen	４．巻 269
２．論文標題 Gromov--Witten theory of quotients of Fermat Calabi-Yau varieties	５．発行年 2021年
３．雑誌名 Memoirs of the American Mathematical Society	６．最初と最後の頁 104p
掲載論文のDOI（デジタルオブジェクト識別子） 10.1090/memo/1310	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Jipeng Cheng and Todor Milanov	４．巻 27
２．論文標題 The extended D-Toda hierarchy	５．発行年 2021年
３．雑誌名 Selecta Mathematica	６．最初と最後の頁 85p
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00029-021-00646-1	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Todor Milanov	４．巻 -
２．論文標題 The phase factors in singularity theory	５．発行年 2019年
３．雑誌名 Advanced Studies in Pure Mathematics	６．最初と最後の頁 295-326
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

1．著者名 Jipeng Cheng and Todor Milanov	4．巻 -
2．論文標題 The 2-component BKP Grassmannian and simple singularities of type D	5．発行年 2019年
3．雑誌名 International Mathematical Research Notices	6．最初と最後の頁 -
掲載論文のDOI（デジタルオブジェクト識別子） 10.1093/imrn/rnz325	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Todor Milanov	4．巻 -
2．論文標題 The total ancestor potential in Singularity Theory	5．発行年 2019年
3．雑誌名 B-model Gromov-Witten theory, Trends in Mathematics	6．最初と最後の頁 539-571
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 ―

1．著者名 Todor Milanov	4．巻 351
2．論文標題 The period map for quantum cohomology of P^2	5．発行年 2019年
3．雑誌名 Advances in Mathematics	6．最初と最後の頁 804-869
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.aim.2019.05.011	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 ―

1．著者名 Milanov Todor、Ruan Yongbin、Shen Yefeng	4．巻 vol. 735(2018)
2．論文標題 Gromov-Witten theory and cycle-valued modular forms	5．発行年 2018年
3．雑誌名 Journal fur die reine und angewandte Mathematik	6．最初と最後の頁 287-315
掲載論文のDOI（デジタルオブジェクト識別子） 10.1515/crelle-2015-0019	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Milanov Todor、Tonita Valentin	4．巻 なし
2．論文標題 K-theoretic Gromov-Witten invariants in genus 0 and integrable hierarchies	5．発行年 2018年
3．雑誌名 Letters in Mathematical Physics	6．最初と最後の頁 1-29
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s11005-018-1075-y	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

〔学会発表〕　計15件（うち招待講演　15件／うち国際学会　13件）

1．発表者名 Todor Milanov
2．発表標題 Fano orbifold lines of type D and integrable hierarchies
3．学会等名 Geometry and Physics seminar, Jiaotong University（招待講演）（国際学会）
4．発表年 2022年

1．発表者名 Todor Milanov
2．発表標題 Frobenius manifolds and vertex operators
3．学会等名 ZOOM lectures at Chinese University of Mining and Technology（招待講演）（国際学会）
4．発表年 2022年

1．発表者名 Todor Milanov
2．発表標題 Reflection vectors for semi-simple Frobenius manifolds
3．学会等名 IBS-CGP, 10th Anniversary Conference（招待講演）（国際学会）
4．発表年 2022年

１．発表者名 Todor Milanov
２．発表標題 Topological recursion for simple singularities
３．学会等名 IBS, Pohang, CGP seminar（招待講演）（国際学会）
４．発表年 2021年

１．発表者名 Todor Milanov
２．発表標題 Integrability in quantum field theory
３．学会等名 Kavli IPMU Colloquium（招待講演）
４．発表年 2021年

１．発表者名 Todor Milanov
２．発表標題 Confluence for the K-theoretic J-function
３．学会等名 Riken iTHEMS seminar（招待講演）
４．発表年 2021年

１．発表者名 Todor Milanov
２．発表標題 Towards Hirota Bilinear Equations in Gromov-Witten theory
３．学会等名 ZOOM lectures at Fudan University（招待講演）（国際学会）
４．発表年 2020年

１．発表者名
Todor Milanov

２．発表標題
Primitive forms and Frobenius structures on the Hurwiotz spaces

３．学会等名
Mirror symmetry and related topics（招待講演）（国際学会）

４．発表年
2019年

１．発表者名
Todor Milanov

２．発表標題
Fano orbifold lines of type D and integrable hierarchies

３．学会等名
3d mirror symmetry and integrable hierarchies（招待講演）（国際学会）

４．発表年
2019年

１．発表者名
Todor Milanov

２．発表標題
Gromov--Witten invariants of Fano orbifold lines of type D and integrable hierarchies

３．学会等名
2nd IBS-CGP workshop on integrable systems and applications（招待講演）（国際学会）

４．発表年
2019年

１．発表者名
Todor Milanov

２．発表標題
Lectures on singularity theory and mirror symmetry

３．学会等名
Current Developments in Geometry（招待講演）（国際学会）

４．発表年
2018年

１．発表者名 Todor Milanov
２．発表標題 The Landau-Ginzburg B-model
３．学会等名 Crossing the walls in enumerative geometry 2018（招待講演）（国際学会）
４．発表年 2018年

１．発表者名 Todor Milanov
２．発表標題 Primitive forms and Frobenius structures on the Hurwitz spaces
３．学会等名 Hodge theory, Stokes phenomenon and applications（招待講演）（国際学会）
４．発表年 2017年

１．発表者名 Todor Milanov
２．発表標題 The period map for quantum cohomology of P^2
３．学会等名 Mathematics day in Sofia 2017（招待講演）（国際学会）
４．発表年 2017年

１．発表者名 Todor Milanov
２．発表標題 The period map for quantum cohomology of P^2
３．学会等名 FJV 2017 Symposium（招待講演）（国際学会）
４．発表年 2017年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

－

6．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

7．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

8．本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関			
中国	IAS at Zhejiang University	CUMT		
韓国	IBS at Pohang University			
米国	University of Oregon	North Carolina State University		