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研究成果の概要（和文）：この研究は幾何学的表現論といわれる代数と幾何をつなぐ数学の分野に属するもので、モジュライ空間と呼ばれる幾何学からあらわれる量子群と呼ばれる代数構造の新たな例を作り出そうとするものです。既存の研究では箙から量子群を作り出すところ、この研究では曲線からホール代数という手法およびその拡張を用い幾何学的な方法で、ベッチ代数、ドラーム代数、ドルボー代数という三つの代数を作り出しました。これらは対応するモジュライ空間の幾何学の対称性を記述しますが、この空間は幾何学の非可換ホッジ理論、また物理のゲージ理論において重要です。このようにこの研究の結果は幾何、代数、物理の深い関係をしめし、更なる研究が望まれます。

研究成果の学術的意義や社会的意義
この研究によってベッチ代数、ドラーム代数、ドルボー代数というまったく新しいタイプの量子群が導入されました。これらはさらに研究する価値があります。これらの代数は幾何学的に導入されましたが、もとになった空間は数え上げ幾何や幾何学的ラングランズプログラムなど数学のいろいろな分野で研究されて来ましたので、それぞれのモジュライ空間の形(トポロジー)に新たな観点を提供します。さらに、これらの代数はアルダイ=ガイオット=立川予想などの、量子物理と幾何学との間の未解決予想の更なる理解に重要であろうと思われます。

研究成果の概要（英文）：The project fits into the realm of geometric representation theory, which stems from synergies between algebra and geometry. The goal of the project was the discovery of new algebraic structures (quantum groups) arising from the study of (moduli) spaces in geometry. We have introduced new quantum groups naturally associated to curves (opposite to those known in the literature, associated to quivers). Our approach to their definition has been geometric, based on the theory of Hall algebras and their refined versions (cohomological, K-theoretical, etc). Specifically, we have defined three algebras: the Betti, the de Rham, and the Dolbeault algebras of a curve. They represent new symmetries arising from the geometry of the corresponding moduli spaces, which play a preeminent role in geometry (e.g., in non-abelian Hodge theory) and in physics (e.g., in gauge theory). Thus, they unlock a new striking connection between geometry, algebra, and physics, which needs to be investigated further.

研究分野：Algebraic geometry, representation theory

キーワード：Quantum groups　Hall algebras　moduli space　moduli stack　Betti shape　De Rham shape　Dolbeault shape　Gauge theory

１．研究開始当初の背景

- Cohomological Hall algebras (à la Schiffmann and Vasserot).

Let Q be a *quiver* (i.e., an oriented graph with a finite number of vertices and oriented arrows) and let **Rep**(Q, **v**) be the stack of representations of Q with fixed dimension **v**. The cotangent stack T***Rep**(Q,**v**) of **Rep**(Q,**v**) is the stack **Rep**(π_Q,**v**) of representations of the so-called *preprojective algebra* π_Q of the quiver Q with fixed dimension **v**. The cohomological Hall algebra CoHA_Q of Q is, as a vector space, the direct sum of (equivariant) Borel-Moore homologies of the T***Rep**(Q,**v**)'s by varying of the **v**'s. The associative product is given as a *convolution product* via the stack parametrizing short exact sequences of representations of the preprojective algebra π_Q.

- Maulik-Okounkov Yangian and Lie algebra.

Maulik and Okounkov has defined another associative algebra associated with an arbitrary quiver. Their construction hinges on the notion of stable envelope to produce a quantum R-matrix, and then on the RTT formalism to define an associative algebra YQ. Taking a quasi-classical limit, they also defined a classical R-matrix and a graded Lie algebra gQ. If Q is of finite type, gQ is the Lie algebra associated with Q, and YQ is the Yangian of the same type. In general, YQ is a deformation of the enveloping algebra of the Lie algebra of polynomial loops gQ[z]. If Q is the one-loop quiver, YQ is the Yangian associated with the affine gl(1). In both mentioned cases, (a positive part of) YQ coincides with CoHAQ; in general, this equality is still an open question and it was proved only that CoHAQ is a subalgebra of (a positive part of) YQ. Okounkov conjectured that the graded dimension of gQ[v] coincides with the Kac polynomial Av(t-1) – this conjecture extends Kac's conjecture (nowadays, theorem) for finite type quivers, relating the dimension of the Lie algebra gQ with the leading term of Av(t-1). Recall that Av(q) counts the number of absolutely indecomposable representations of Q over a finite field with q elements. The way to address Okounkov's conjecture pursued by Schiffmann and Vasserot was via cohomological Hall algebras.

- Kac polynomial of a curve and Higgs sheaves.

Schiffmann has introduced the analog of Kac polynomial for a curve: he proved that there exists a unique polynomial Ag,r,d such that its evaluation at the Weil numbers of a smooth projective curve of genus g over a finite field with q elements is equal to the number of absolutely indecomposable vector bundles of rank r and degree d.

A "key question" to ask is the curve analog of Okounkov's conjecture, which is the following: given a smooth projective curve X of genus g, is there a graded Lie algebra gX such that the dimension of the graded piece gX[r,d] is given by Ag,r,d?

２．研究の目的

The present project aims to elucidate the "key question" by using the theory of cohomological Hall algebras, by developing the following points:

a. Define a cohomological Hall algebra CoHAX for a curve X.
b. Study the genus zero case: provide an explicit description of CoHAX for X=P1 to understand if it is a Yangian (or at least it contains a Yangian as a subalgebra).

c. Define a filtration of CoHAX compatible with the algebra structure, such that the graded algebra is generated by primitive elements.

d. Study the Lie algebra generated by primitive elements and compare its (graded) dimension with the Kac polynomial of the curve.

３．研究の方法

(a) (Dolbeault) Cohomological Hall algebra of X. Let X be a smooth projective complex curve of genus g. Denote by Coh(X,r,d) the stack of coherent sheaves on X of rank r and degree d. This stack plays the role of the stack Rep(Q,v) in the curve case, therefore the cohomological Hall algebra CoHAX of X is, as a vector space, the direct sum of (equivariant) Borel-Moore homologies of the stacks T*Coh(X,r,d)'s by varying of the ranks and degrees. T*Coh(X,r,d) coincides with the stack Higgs(X,r,d) parametrizing pairs (E, f), where E is a coherent sheaf on X of rank r and degree d, and f is a morphism between E and the tensor product of E with ΩX, where ΩX is the canonical bundle of X. These pairs are called Higgs sheaves on X. Analogously, one can introduce the stack Higgs(X,r1,d1,r2,d2) parametrizing short exact sequences 0 —> (E1, f1) —> (E, f) —> (E2, f2) —> 0 of Higgs sheaves on X, with E1 of rank r1 and degree d1, and E2 of rank r2 and degree d2. There are two natural maps

p(r1,d1;r2,d2): Higgs(X,r1,d1,r2,d2) —> Higgs(X,r1,d1) x Higgs(X,r2,d2), and
q(r1,d1;r2,d2): Higgs(X,r1,d1,r2,d2) —> Higgs(X,r1+ r2,d1+d2).

The product in Borel-Moore homology should be defined as the composition of the pullback with respect to p(r1,d1;r2,d2) with the pushforward with respect to q(r1,d1;r2,d2). The definition of this product and the proof of its associativity are more involved than in the quiver case, since the stacks I am considering are not anymore of finite type as it was in the quiver case.

(b) Betti, de Rham, Dolbeault Cohomological Hall algebra of X. The approach described in (a) is doable only for the Higgs case. On the other hand, in the curve case, we can consider other two moduli stacks naturally attached to a curve: the de Rham moduli stack, which parametrizes vector bundles on X with flat connections, and the Betti moduli stack, which parametrizes finite-dimensional representations of the fundamental group of X. In order to define algebras associated to these all these stacks in a unified way, we use Simpson's theory of shapes of varieties and natural derived enhancements of the moduli stacks mentioned above. This provides us a way to construct not only the Betti, de Rham, and Dolbeault cohomological Hall algebra of X, but also to categorify the construction at the level of dg categories of coherent complexes over the derived enhancements. Moreover, the use of shapes provides a way to compare the 3 algebras and obtain versions of the Riemann-Hilbert correspondence and the non-abelian Hodge correspondence for these algebras.

(c) Genus zero case. It is expected that the Dolbeault cohomological Hall algebra of the projective line is the affine Yangian of type A_1. This is geometrically justified by the derived equivalence between the derived category of sheaves on the projective line and the derived category of modules of the Kronecker quiver, or, equivalently, by the derived McKay correspondence for the type A_1 singularity. The Dolbeault algebra should provide a half of the affine Yangian which is different from the half provided by the quiver construction. Geometrically this is justified by the fact that the above derived equivalences are not exact. A study of the relationship between the Dolbeault algebra and the mentioned derived equivalence should enlighten the description as a Yangian.

４．研究成果

The main results of the project are the following ones.

1. The introduction of the Betti, de Rham, Dolbeault cohomological Hall algebras of a smooth projective curve.
2. Categorified versions of the above algebras have been provided as symmetric monoidal structures on the dg categories of coherent complexes on the Betti, de Rham, Dolbeault (derived) moduli stacks.
3. The above algebras (and their categorifications) are related by Hall algebra versions of the Riemann-Hilbert correspondence and the non-abelian Hodge correspondence for these algebras.
4. In the projective line case, the semistable Dolbeault algebra has been described explicitly (and more generally I obtained a characterization of the semistable cohomological Hall algebra of the resolution of the A_N singularity). (This latter part was concluded in April 2020 because of the lockdown of Italy, France, and USA due to the COVID-19 Pandemic.)

These results have appeared on:

- Duiliu-Emanuel Diaconescu, Mauro Porta, Francesco Sala. McKay correspondence, cohomological Hall algebras and categorification. arXiv:2004.13685
- Mauro Porta, Francesco Sala. Two-dimensional categorified Hall algebras. arXiv:1903.07253
- Francesco Sala, Olivier Schiffmann. Cohomological Hall algebra of Higgs sheaves on a curve. arXiv:1801.03482 (also, it appeared on *Algebraic Geometry*, volume no. 7, issue no. 3, pages 346-376, 2020.)

The understanding of the full Dolbeault algebra is a subject of a current investigation: I am investigating the relation between the cohomological Hall algebra of the resolution of the A_N singularity (for N=1, one has the Dolbeault algebra) and the braid group action coming from the derived category of sheaves on the resolution of the A_N singularity. (This approach is analogous to that of Beck for quantum enveloping algebras.)

Finally, the construction of the categorified version of the algebras was not planned in the proposal, but it arisen naturally by using derived algebraic geometry. An important consequence of the construction is the dg coherent categorification of (a positive half of) the elliptic Hall algebra. It will be subject of a future investigation the relationship between my dg coherent categorification, the coherent categorification of Shan-Varagnolo-Vasserot recently introduced, and Khovanov-Lauda e Rouquier categorification.

5．主な発表論文等

〔雑誌論文〕　計4件（うち査読付論文　4件／うち国際共著　4件／うちオープンアクセス　4件）

1．著者名	4．巻
F. Sala, O. Schiffmann	no.7,issue no.3
2．論文標題	5．発行年
Cohomological Hall algebra of Higgs sheaves on a curve	2020年
3．雑誌名	6．最初と最後の頁
Algebraic Geometry	346～376
掲載論文のDOI（デジタルオブジェクト識別子）	査読の有無
10.14231/AG-2020-010	有
オープンアクセス	国際共著
オープンアクセスとしている（また、その予定である）	該当する

1．著者名	4．巻
F. Sala, O. Schiffmann	-
2．論文標題	5．発行年
Fock Space Representation of the Circle Quantum Group	2019年
3．雑誌名	6．最初と最後の頁
International Mathematics Research Notices,IMRN	-
掲載論文のDOI（デジタルオブジェクト識別子）	査読の有無
10.1093/imrn/rnz268	有
オープンアクセス	国際共著
オープンアクセスとしている（また、その予定である）	該当する

1．著者名	4．巻
Sala Francesco、Schiffmann Olivier	no.5,Art.77
2．論文標題	5．発行年
The circle quantum group and the infinite root stack of a curve	2019年
3．雑誌名	6．最初と最後の頁
Selecta Mathematica	86
掲載論文のDOI（デジタルオブジェクト識別子）	査読の有無
10.1007/s00029-019-0521-8	有
オープンアクセス	国際共著
オープンアクセスとしている（また、その予定である）	該当する

1．著者名	4．巻
Eyssidieux Philippe、Sala Francesco	27
2．論文標題	5．発行年
Instantons and framed sheaves on Kahler Deligne-Mumford stacks	2018年
3．雑誌名	6．最初と最後の頁
Annales de la faculte' des sciences de Toulouse Mathe'matiques	599～628
掲載論文のDOI（デジタルオブジェクト識別子）	査読の有無
10.5802/afst.1579	有
オープンアクセス	国際共著
オープンアクセスとしている（また、その予定である）	該当する

〔学会発表〕　計21件（うち招待講演　20件／うち国際学会　21件）

1．発表者名 Francesco Sala
2．発表標題 Cohomological Hall algebras of a curve
3．学会等名 Algebraic and Arithmetic Geometry Seminar, University of Pisa (Italy)（招待講演）（国際学会）
4．発表年 2020年

1．発表者名 Francesco Sala
2．発表標題 Cohomological Hall algebras of a curve
3．学会等名 Algebra and Geometry Seminar, University of Bologna (Italy)（招待講演）（国際学会）
4．発表年 2020年

1．発表者名 Francesco Sala
2．発表標題 Continuum Kac-Moody algebras
3．学会等名 Algebra and Representation Theory, University of Roma “Tor Vergata” (Italy)（招待講演）（国際学会）
4．発表年 2020年

1．発表者名 Francesco Sala
2．発表標題 Continuum quantum groups
3．学会等名 XXI Congresso dell'Unione Matematica Italiana, Lie Theory Section（招待講演）（国際学会）
4．発表年 2019年

１．発表者名
Francesco Sala

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Continuum quantum groups

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Cohomological Hall algebras of a curve

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３．学会等名 XXI Congresso dell'Unione Matematica Italiana, Sezione "Teoria di Lie". September 2 - 6, 2019, University of Pavia, Italy.（招待講演）（国際学会）
４．発表年 2019年

１．発表者名 Francesco Sala
２．発表標題 Categorification of two-dimensional cohomological Hall algebras
３．学会等名 Geometry, Symmetry, and Physics Seminar. Rutgers University, USA. April 11, 2019.（招待講演）（国際学会）
４．発表年 2019年

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２．発表標題 Continuum Lie algebras and continuum quantum groups
３．学会等名 Geometry, Physics, and Representation Theory Seminar. Northeastern University, USA. April 4, 2019.（招待講演）（国際学会）
４．発表年 2019年

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２．発表標題 Categorification of two-dimensional cohomological Hall algebras
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２．発表標題 Topology of moduli spaces via representation theory
３．学会等名 Hong Kong Geometry Colloquium. November 3, 2018.（招待講演）（国際学会）
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４．発表年
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３．学会等名
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４．発表年
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４．発表年 2018年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

Sala Francesco's personal webpage http://www.salafrancesco.altervista.org/research.html

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