様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

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研究課題名（和文）Development of Viscosity and Variational Techniques for the Analysis of Moving Interfaces

研究課題名（英文）Development of Viscosity and Variational Techniques for the Analysis of Moving Interfaces

研究代表者

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研究成果の概要（和文）：境界面の概念は、物理学、材料科学、生物学の多くの重要な学術問題に使われ、異なる移動する領域を分離します。境界面は衝突したり分割されることがあり、そのような時点を越えた境界面の運動の適切な継続は難しいです。結晶成長のモデルの弱解の理論に貢献し、動力が働くクリスタライン平均曲率流というモデルの解の存在性と一意性を示しました。これにより、このモデルを雪の結晶の成長など、より現実的な状況で利用することが可能になります。また、結晶の所定の体積を持つ関連する問題も研究しました。そして、材料の多数の転位がどのように移動するかを研究しました。それらの動きは、材料が変形にどのように反応するかを決定します。

研究成果の学術的意義や社会的意義
研究した数学問題は、物理学、材料科学、生物学、人口動態などのさまざまな分野の重要なモデルに使われます。得られた理論的結果は、これらの問題の特徴についての理解を深め、コンピューター上で解を求めるための精度が高い数値解法を開発するための重要な基盤となります。これらのモデルを結晶成長、材料の塑性、土壌中の水の流れなどの応用研究に活用していただければ幸いです。

研究成果の概要（英文）：Interfaces that separate different moving regions are a feature of many important problems in physics, material science and biology. The mathematical understanding of equations arising in such problems is crucial for developing accurate numerical methods to find a solution on a computer. The interfaces might collide or split, and a proper continuation of solution past such times is mathematically challenging. We contributed to the theory of generalized solutions of a model of crystal growth, showing existence and uniqueness of a new notion of solution of the so called crystalline mean curvature flow with space dependent forcing, which will allow the use of this model in more realistic situation like the growth of snow crystals. We also studied a related problem with a prescribed volume of the crystal. We used similar mathematical tools to study how a large number of dislocations in a material move. Their movement governs how the material will respond to deformations.

研究分野：関数方程式

キーワード：generized solutions　crystal growth　mean curvature　annihilation　comparison principle

様　式　C－１９、F－１９－１、Z－１９（共通）

１．研究開始当初の背景

An important feature of many problems in modelling is a moving boundary: a sharp interface between two or more regions across which physical or other quantities change abruptly. For example, the surface of a growing crystal or the boundary of the wet region in the flow of liquid through a porous medium. The mathematical treatment of such interfaces is challenging due to their singular nature and it is a subject of ongoing research. Moreover, numerical solution techniques must be developed to specifically treat such interfaces and their singularities caused for instance by topological changes like merging, pinch off, collisions etc. One of the crucial steps in successfully finding a reasonable numerical solution is finding the correct meaning of a solution and the possibly unique way of how to continue a solution passed an appearance of singularity in the solution: the rigorous notion of a generalized solution.

２．研究の目的

The goal of this research was to build on the recent developments of mathematical analysis of generalized solutions for important problems with a free boundary:

(1) Crystalline mean curvature flow: This is one of the popular mathematical models of the growth of small crystals. It describes the surface evolution as the one that decreases the surface energy of the crystal the "fastest" in a certain mathematical sense (Figure 1). When the surface energy is combined with a potential energy, we get the forced mean curvature flow.

(2) Hele-Shaw type problems: These describe the pressure-driven evolution of a fluid with a free surface. In recent years they were used successfully to describe a growth of a tumor tissue and a crowd motion, increasing the number of their applications.

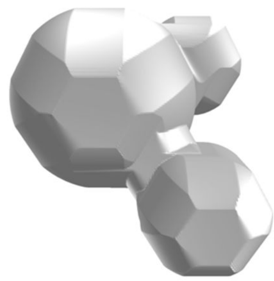

Figure 1

We set to advance the understanding of the generalized solutions of the underlying nonlinear partial differential equations that appear in these types of models. Their common theme is the presence of both a variational structure (the system they describe has an energy that is decreasing), and a comparison principle structure (if two solutions are ordered at some initial time, they stay ordered). Using a combination of these can provide deeper insight into the behavior of the solutions passed singularities.

３．研究の方法

This project pursued multiple directions of research in the proposed problems and other closely related ones. We collaborated with overseas and local mathematicians. The joint work was allowed by my participation in numerous conferences and short research visits of my collaborators. I also had a chance to host invited researchers to share ideas and to consider possible new research directions. One of the important aspects of the abstract analysis is to inform the development of practical numerical schemes for the problems, and some of the numerical work was used to motivate further directions of theoretical research with the help of doctoral students.

４．研究成果

(1) We made significant progress in the theory of a type of generalized solutions, the viscosity solutions, for the crystalline mean curvature flow (MCF) [4]. The notion of viscosity solutions is based on the comparison structure of the problem. Our main contribution in this project was to establish the existence and uniqueness of solutions when a forcing term that depends on space variable is added to the equation. For example, in the model of growth of snow crystals, the forcing is caused by temperature on the crystal surface, in combination with the concentration of water vapor in the

environment surrounding the crystal: the crystal grows faster at lower temperatures and higher concentrations. Since the surface energy of the crystal is smaller on parts of the surface that are parallel to one of the sides of a hexagonal prism, the crystal develops flat parts called facets in these directions, even when there are small differences in the forcing along the facet. However, if the differences in the forcing across a facet are sufficiently large, which often happens at the exposed corners, facets break there and a complicated pattern of new crystalline facets develops. Snowflakes are results of such complicated process of facet breaking and growth. Capturing this effect in the mathematical analysis is rather nontrivial. We needed to use the variational (energy) structure of the problem to understand how the forcing interplays with the tendency to decrease the surface energy (the so called crystalline mean curvature) and create larger flat facets yet allow these facets to be broken when required by the energy structure. This allowed us to establish the comparison principle for these problems (yielding uniqueness of solutions), and we were able to construct the unique solution by an approximation by problems with smoothed (regularized) surface energy, whose mathematical theory is well-established. The compatibility with the classical smooth surface energy problem hints that our notion of generalized solutions is natural. This work is an important step in the direction of coupling the mean curvature flow with a forcing that itself is given by a solution of a Stefan type problem modelling the diffusion of the vapor and conduction of the heat of crystallization. The analysis appears to be rather challenging so in the upcoming Kakenhi project our goal is to develop an efficient numerical method for the crystalline MCF. One of the important unresolved issues is the question of how to correctly prescribe data on the boundary of a domain. The nonlocal nature of the crystalline mean curvature makes this a rather delicate issue.

(2) Due to the above work, the theory of viscosity solutions can be now considered quite complete for the crystalline MCF with purely crystalline anisotropy and a forcing, without boundary. Since there were other recent developments in the study of these problems from the point of minimizing movements, we published an extensive survey of the recent results, with extra comments, examples and worked-out proofs [1].

(3) Simplified models of crystal growth often prescribe the volume of the crystal as a given time-dependent quantity. We therefore considered the crystalline MCF with a prescribed volume constraint [2]. This constraint manifests itself as a nonlocal forcing term. The nonlocal nature is one of the main challenges of this modification: the problem does not have a comparison principle. This severely limits the analytical tools to study the problem and there are very few results on the existence and uniqueness of solutions in a general situation. We considered the well-posedness of the problem when the initial data has an additional natural geometric reflection property with respect to the reflection symmetries of the crystal, resembling starshapeness of the initial crystal. In this somewhat restricted setting, we were able to show existence of a notion of solutions of the crystalline MCF with volume constraint, and establish a local Lipschitz in space, local Holder in time regularity of the solution. The uniqueness of solutions, as well as the existence for general initial data is still an open problem.

(4) We have achieved further progress in understanding of the behavior of problems with a free boundary interacting with a highly oscillating medium, the so-called homogenization of Hele-Shaw problems, describing for example the evolution of water surface in a porous medium like sand or soil. In this work [5], we considered a medium whose properties vary on a very small scale both in space and in time. A possible motivation is a day-night or seasonal variation in the properties of soil. A previous theoretical result on the homogenization of this problem, which studies the average effect of the variations, suggested a very complicated behavior of the average free surface velocity that depends on the direction of travel. This is a unique consequence of the interplay of the space and time variations that make some directions of movement more favorable. However, there is no explicit formula for the average velocity. To address this, we developed an efficient numerical method in two space dimensions based on the theoretical work to find the average velocity of the free boundary when it is being perturbed by a space-time dependent irregularity. We have observed that the surface might develop flat parts resembling the facets of a crystal. This appears to be one of the first suggestions that a dynamic problem of this kind can exhibit facets

stable in time. With a doctoral student, we are currently pursuing improvements of this numerical method to make it more practical to handle more general situations and three-dimensional settings.

(5) We were able to adapt the techniques pursued in this project to a somewhat related problem that models the motion of dislocation lines in a crystalline material [3]. We considered a one-dimensional situation where the dislocation lines are all parallel and are located in a single plane. Therefore, they can be described by points in one dimension. Due to the elastic stress that the dislocation lines cause in the material, they are attracted or repelled by other dislocation lines, depending on the mutual direction (represented by a charge +1 or -1) of the dislocations. When multiple lines collide at a given place and time in a process called annihilation, either the dislocations completely cancel out, or a single dislocation line remains so that the total charge is preserved. The problem has a clear variational structure where the lines are moving in such a way that they decrease the elastic stress in the “fastest” possible way. However, this problem exhibits a hidden comparison principle. The points representing the dislocation lines can be described as level sets (sets of constant value) of an auxiliary function, and this function then satisfies a nonlinear partial differential equation with a nonlocal operator. Since a comparison principle holds for this problem, this allows the use of comparison principle-based tools to study this problem. Our main results addressed the many-particle limit of this problem: we rigorously proved that when the number of dislocations is large, their evolution can be approximately described by a limiting nonlinear partial differential equation for a dislocation density function.

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5．主な発表論文等

〔雑誌論文〕　計5件（うち査読付論文　5件／うち国際共著　2件／うちオープンアクセス　4件）

1．著者名 Kim Inwon、Kwon Dohyun、Pozar Norbert	4．巻 online
2．論文標題 On volume-preserving crystalline mean curvature flow	5．発行年 2021年
3．雑誌名 Mathematische Annalen	6．最初と最後の頁 1～42
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00208-021-02286-4	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Giga Yoshikazu、Pozar Norbert	4．巻 1
2．論文標題 Viscosity solutions for the crystalline mean curvature flow with a nonuniform driving force term	5．発行年 2020年
3．雑誌名 SN Partial Differential Equations and Applications	6．最初と最後の頁 1～26
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s42985-020-00040-0	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

1．著者名 Palupi Irma, Pozar Norbert	4．巻 62
2．論文標題 An efficient numerical method for estimating the average free boundary velocity in an inhomogeneous Hele-Shaw problem	5．発行年 2019年
3．雑誌名 The Science Reports of Kanazawa University	6．最初と最後の頁 69～86
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

1．著者名 Giga Yoshikazu、Pozar Norbert	4．巻 12
2．論文標題 Motion by crystalline-like mean curvature: A survey	5．発行年 2022年
3．雑誌名 Bulletin of Mathematical Sciences	6．最初と最後の頁 1～68
掲載論文のDOI（デジタルオブジェクト識別子） 10.1142/S1664360722300043	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 －

１．著者名 van Meurs Patrick、Peletier Mark A.、Pozar Norbert	４．巻 246
２．論文標題 Discrete-to-Continuum Convergence of Charged Particles in 1D with Annihilation	５．発行年 2022年
３．雑誌名 Archive for Rational Mechanics and Analysis	６．最初と最後の頁 241～297
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s00205-022-01812-1	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

〔学会発表〕　計14件（うち招待講演　12件／うち国際学会　8件）

１．発表者名 Pozar Norbert
２．発表標題 Incompressible limit of the porous medium equation with a drift
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１．発表者名 Pozar Norbert
２．発表標題 Crystalline mean curvature flow with a volume constraint
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４．発表年 2021年

１．発表者名 Norbert Pozar
２．発表標題 Forcing in the crystalline mean curvature flow
３．学会等名 Surface and Interface Dynamics II（招待講演）（国際学会）
４．発表年 2020年

1．発表者名 Norbert Pozar
2．発表標題 Forcing and volume constraint in the crystalline mean curvature flow
3．学会等名 2020 KMS Annual Meeting（招待講演）（国際学会）
4．発表年 2020年

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2．発表標題 Forcing and volume constraint in the crystalline mean curvature flow
3．学会等名 大岡山談話会（招待講演）
4．発表年 2020年

1．発表者名 Norbert Pozar
2．発表標題 Self-similar solutions of the crystalline mean curvature flow
3．学会等名 Geometric Aspects of Solutions to Partial Differential Equations (RIMS)（招待講演）
4．発表年 2019年

1．発表者名 Norbert Pozar
2．発表標題 Viscosity approach to the crystalline mean curvature flow and its applications
3．学会等名 ICIAM2019（国際学会）
4．発表年 2019年

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Norbert Pozar

２．発表標題
Viscosity approach to the crystalline mean curvature flow and the crystalline algorithm

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Kyushu Functional Equation Seminar（招待講演）

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Large-time behavior of the anisotropic Stefan problem in nonuniform media

３．学会等名
The 12th AIMS Conference on Dynamical Systems, Differential Equations and Applications（招待講演）（国際学会）

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2018年

１．発表者名
Norbert Pozar

２．発表標題
Incompressible limit of the porous medium equation with a drift

３．学会等名
The 12th AIMS Conference on Dynamical Systems, Differential Equations and Applications（招待講演）（国際学会）

４．発表年
2018年

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４．発表年 2023年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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６．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

７．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

８．本研究に関連して実施した国際共同研究の実施状況

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オランダ	Eindhoven University of Technology			