様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９　（共通）

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研究課題名（和文）Multi-aspects of beta ensembles and related random matrix models

研究課題名（英文）Multi-aspects of beta ensembles and related random matrix models

研究代表者
Ｔｒｉｎｈ　Ｋｈａｎｈ・Ｄｕｙ（Trinh, Khanh Duy）

早稲田大学・理工学術院・准教授（任期付）

研究者番号：００７２６１２７

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研究成果の概要（和文）：本研究の対象は実ランダム行列におけるベータアンサンブル、特に3つの古典ベータアンサンブル（ガウシアンベータアンサンブル、ベータラゲールアンサンブル、ベータヤコビアンサンブル）である。一般のベータアンサンブルの高温極限において普遍のローカル統計の結果を得た。また古典ベータアンサンブルでは、高温極限において固有値の経験分布に対する大数の法則と中心極限定理を証明し、その収束先はエルミート多項式、ラゲール多項式とヤコビ多項式にそれぞれ関連することを示した。

研究成果の学術的意義や社会的意義
We have developed new approaches to study beta ensembles, especially the three classical beta ensembles. By those approaches, we can completely describe the global and the local asymptotic behavior of beta ensembles in a high temperature regime.

研究成果の概要（英文）：We study beta ensembles on the real line with focusing on the three classical beta ensembles (Gaussian beta ensembles, beta Laguerre ensembles and beta Jacobi ensembles). In a high temperature regime, we show a universality result at the bulk, that is, around any fixed reference energy, the local statistics converges in distribution to a homogeneous Poisson point process. For the three classical beta ensembles, we completely describe the global behavior, that is, two fundamental results on the convergence to a limit of the empirical distribution (law of large numbers) and Gaussian fluctuations around the limit (central limit theorem). We flexibly use tools from probability theory, spectral theory, theory of orthogonal polynomials and stochastic analysis. The limiting measure in a high temperature regime is related to associated Hermite polynomials (Gaussian case), associated Laguerre polynomials (Laguerre case) and associated Jacobi polynomials (Jacobi case).

研究分野：probability theory

キーワード：beta ensembles　high temperature regime　orthogonal polynomials　Gaussian fluctuations

様　式　Ｃ－１９、Ｆ－１９－１、Ｚ－１９（共通）

１．研究開始当初の背景

Beta ensembles are objects in random matrix theory, statistical mechanics, potential theory and spectral theory. Among them, three classical beta ensembles on the real line are now realized as eigenvalues of certain random tridiagonal matrices. The parameter beta regarded as the inverse temperature is usually assumed to be fixed. Problems with beta varying have been investigated for some specific beta ensembles recently, leading to some crossover results. This research aims to establish new spectral properties and to provide universal approaches to deal with even the case of beta varying.

２．研究の目的

This research aims to establish new spectral properties of beta ensembles, in general, and of the three classical beta ensembles and their related random matrix models, in particular. It focuses on universal methods which can deal with the case of varying parameter beta as well.

３．研究の方法

The idea is to flexibly use tools from spectral theory, probability theory and potential theory together with an analytic approach based on the joint density.

４．研究成果

This research studies spectral properties of beta ensembles and related random matrix models in case the inverse temperature beta is allowed to vary with the system size. In what follows, a **high temperature regime** is the case where the temperature parameter (the reciprocal of beta) is proportional to the system size. We obtain the following results.

(1) **Gaussian beta ensembles.** They are the most studied random matrix model. The convergence of the empirical distribution of the eigenvalues to a limit and Gaussian fluctuations around the limit have been established even when the parameter beta is allowed to vary as the system size tends to infinity. In a high temperature regime, the empirical distribution converges weakly to a probability measure of associated Hermite polynomials, almost surely. Gaussian fluctuations around the limit can be established by using the random tridiagonal matrix model.

We study dynamical versions of the three classical beta ensembles (Gaussian beta ensembles, beta Laguerre ensembles and beta Jacobi ensembles), that is, stochastic processes called beta Dyson's Brownian motions, beta Laguerre processes and beta Jacobi processes. We develop a moment method at the process level to solve two fundamental problems: the convergence of the empirical measure process to a limiting process, and fluctuations around the limit. This approach yields several new results on the proves level. In addition, it provides a natural way to explain the appearance of orthogonal polynomials in the problem of Gaussian fluctuations around the limit.

Back to the Gaussian case: in a high temperature regime, our new moment approach at the process level yields a fundamental result on Gaussian fluctuations: by taking primitives of associated Hermite polynomials, Gaussian limits are independent.

(2) **Beta Laguerre ensembles.** We completely describe the global asymptotic behavior of the empirical distribution, that is, the convergence to a limit distribution and Gaussian fluctuations around the limit. Beta Laguerre ensembles are generalizations of the distribution of the eigenvalues of Wishart matrices or Laguerre matrices, two types of random matrices in statistics, in terms of the joint density. They are now realized as eigenvalues of a random tridiagonal matrix model. For the proof, we make use of the random matrix model and extend some ideas used in the case of Gaussian beta ensembles.

In a high temperature regime, we show that the empirical distributions converge weakly to a limiting measure which is the probability measure of associated Laguerre polynomials (Model II), almost surely. We also establish a dynamical version of that result. Namely, consider beta Laguerre processes in a high temperature regime, we show that their empirical measure processes converge to a limiting process (in probability). For the proof, we develop a moment method at the process level. The key ideas are: (i) each moment process of the empirical measure processes can be shown to converge to a deterministic process by induction, and (ii) under some additional mild conditions, the limiting moment processes determine the limiting measure uniquely, and thus, the convergence of the empirical measure processes follows. Similar to the Gaussian case, Gaussian fluctuations at the process level have also been studied.

(3) **Beta Jacobi ensembles.** We extend the dynamical approach to study beta Jacobi ensembles and beta Jacobi processes at a high temperature regime. For the result, we obtain a new model of associated Jacobi polynomials, named Model III. We also establish a fundamental result on Gaussian fluctuations around the limit, or central limit theorems involving orthogonal polynomials. The Jacobi case is technically more difficult than the Gaussian case and the Laguerre case. Detailed arguments involve: result on the freezing regime in which the system size is fixed while the parameter beta tends to infinity, duals of Jacobi polynomials, joint convergence of stochastic processes and their initial data. Two important new ideas here are: (i) dealing with stationary beta Jacobi processes whose stationary distribution is nothing but the corresponding beta Jacobi ensembles; and (ii) showing that in the limit, the initial conditions are independent of the rest of the process.

(4) **Beta ensembles on the real line with generic potentials.** In a high temperature regime, we show that the local statistics around any fixed reference energy converges to a homogeneous Poisson point process. We prove the Poisson statistics by analyzing the joint density with the help of some estimates from the theory of large deviation principle.

５．主な発表論文等

〔雑誌論文〕　計6件（うち査読付論文　6件／うち国際共著　5件／うちオープンアクセス　2件）

１．著者名 Nakano Fumihiko、Trinh Hoang Dung、Trinh Khanh Duy	４．巻 未定
２．論文標題 Limit theorems for moment processes of beta Dyson ' s Brownian motions and beta Laguerre processes	５．発行年 2023年
３．雑誌名 Random Matrices: Theory and Applications	６．最初と最後の頁 未定
掲載論文のDOI（デジタルオブジェクト識別子） 10.1142/S2010326323500053	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Can Van Hao、Trinh Khanh Duy	４．巻 27
２．論文標題 Random connection models in the thermodynamic regime: central limit theorems for add-one cost stabilizing functionals	５．発行年 2022年
３．雑誌名 Electronic Journal of Probability	６．最初と最後の頁 1-40
掲載論文のDOI（デジタルオブジェクト識別子） 10.1214/22-EJP759	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

１．著者名 Trinh Hoang Dung、Trinh Khanh Duy	４．巻 185
２．論文標題 Beta Jacobi Ensembles and Associated Jacobi Polynomials	５．発行年 2021年
３．雑誌名 Journal of Statistical Physics	６．最初と最後の頁 1--15
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s10955-021-02832-z	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

１．著者名 Trinh Hoang Dung、Trinh Khanh Duy	４．巻 58(2)
２．論文標題 Beta Laguerre ensembles in global regime	５．発行年 2021年
３．雑誌名 Osaka J. Math.	６．最初と最後の頁 435~450
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 有
オープンアクセス オープンアクセスとしている（また、その予定である）	国際共著 該当する

1．著者名 Trinh Hoang Dung、Trinh Khanh Duy	4．巻 136
2．論文標題 Beta Laguerre processes in a high temperature regime	5．発行年 2021年
3．雑誌名 Stochastic Processes and their Applications	6．最初と最後の頁 192～205
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.spa.2021.03.002	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 該当する

1．著者名 Fumihiko Nakano、Khanh Duy Trinh	4．巻 179
2．論文標題 Poisson Statistics for Beta Ensembles on the Real Line at High Temperature	5．発行年 2020年
3．雑誌名 Journal of Statistical Physics	6．最初と最後の頁 632～649
掲載論文のDOI（デジタルオブジェクト識別子） 10.1007/s10955-020-02542-y	査読の有無 有
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

〔学会発表〕　計9件（うち招待講演　3件／うち国際学会　4件）

1．発表者名 Khanh Duy Trinh
2．発表標題 Orthogonal polynomials and classical beta ensemble
3．学会等名 Annual Conference on Probability and Related Topics（招待講演）（国際学会）
4．発表年 2022年

1．発表者名 Khanh Duy Trinh
2．発表標題 Mean-field limit for stochastic processes related to classical beta ensembles
3．学会等名 Interplay of partial differential equations and stochastic processes（招待講演）（国際学会）
4．発表年 2022年

１．発表者名
Khanh Duy Trinh

２．発表標題
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３．学会等名
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４．発表年
2022年

１．発表者名
Khanh Duy Trinh

２．発表標題
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４．発表年
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１．発表者名
Khanh Duy Trinh

２．発表標題
Global spectrum fluctuations for Gaussian beta ensembles

３．学会等名
Danwakai, Tohoku University

４．発表年
2021年

１．発表者名
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２．発表標題
On beta Laguerre ensembles at varying temperature

３．学会等名
Japanese-German Open Conference on Stochastic Analysis 2019（国際学会）

４．発表年
2019年

１．発表者名
Khanh Duy Trinh

２．発表標題
Local statistics for beta ensembles at high temperature

３．学会等名
The 18th Symposium Stochastic Analysis on Large Scale Interacting Systems

４．発表年
2019年

１．発表者名
Khanh Duy Trinh

２．発表標題
On Wishart processes

３．学会等名
One-day Symposium: Hydrodynamic limit and related topics

４．発表年
2019年

１．発表者名
Khanh Duy Trinh

２．発表標題
On the moment method for beta Wishart processes

３．学会等名
Spectra of Random Operators and Related Topics

４．発表年
2020年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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６．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

７．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

８．本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関