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研究課題名（和文）Hamilton-Jacobi equations on metric measure spaces

研究課題名（英文）Hamilton-Jacobi equations on metric measure spaces

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研究成果の概要（和文）：本研究では，主に三つの課題について研究を行なった。第１部では、距離空間におけるeikonal方程式について，新しい概念であるMonge解を導入し，既存の解の概念の同値性を示しました。さらに、そのMonge解を用いて不連続なeikonal方程式についても研究した。第２部では、距離空間における Hamilton-Jacobi方程式のゲーム論的解釈を確立した．時間連続ゲームを構築し、その値関数を用いた明示的な解の表現公式を与えた。第３部では、Heisenberg群における凸関数と準凸関数の性質について考察し，偏微分方程式に基づくアプローチをよる新たな特徴づけを導出した．

研究成果の学術的意義や社会的意義
Although several notions of viscosity solutions to the HJ equations on metric spaces have been introduced, our research reveals intrinsic connections between numerous results on HJ equations in general settings and has great potential to be applied in other fields.

研究成果の概要（英文）：First, showing the equivalence of solutions of eikonal equations in metric spaces by introducing a new simple notions called Monges solution. Exploiting the notion of Monge solution and using it to study discontinuous eikonal equations in metric measure spaces which produces even new results in the Euclidean spaces. Second, constructing a two-person continuous-time game in a geodesic space and show that the value function is the unique solution of the Hamilton-Jacobi equation. Our result develops, in a general geometric setting, the classical connection between differential games and the viscosity solutions to possibly nonconvex Hamilton-Jacobi equations. Third, using first-order and second-order PDE-based approaches to study the horizontally quasiconvex (h-quasiconvex for short) functions in the Heisenberg group and apply the characterizations to construct h-quasiconvex envelope and study the h-convexity preserving property for horizontal curvature flow in the Heisenberg group.

研究分野：Analysis on metric spaces

キーワード：Eikonal equation　viscosity solution　metric spaces　Heisenberg group　differential games　convexity

１．研究開始当初の背景

Hamilton-Jacobi equations in the Euclidean spaces are widely applied in various fields such as optimal control, geometric optics, computer vision and image processing. It is well known that the HJ equations in general do not admit classical solutions. The viscosity solution theory provides a very successful framework for studying the well-posedness including existence, uniqueness and stability of the solutions and achieves fruitful results solving questions raised from diverse fields.

In the recent decade, motivated by the developments of mean field games, optimal transport and other related areas, there emerges a pressing need to generalize the HJ equations to general metric spaces. Based on distinct perspectives from the viscosity solution theory in the Euclidean spaces, several different notions of viscosity solutions to the HJ equations on metric spaces have been proposed in different contexts. This leads to the natural questions about the equivalence of these solutions in general spaces and whether there is a unified framework for the well-posedness of a more general class of the HJ equations.

２．研究の目的

The purpose of the project is to enhance the understanding of solutions to the first order differential equations in metric spaces. The first aim is to answer the following key questions of our projects: How can we compare these different approaches to viscosity solutions to the HJ equations on metric spaces? The second aim is to study the properties of various HJ equations in different geometric settings and explore the applications.

３．研究の方法

The methods employed in the projects include the theory of viscosity solutions in the Euclidean spaces, optimal control interpretation and differential game interpretation of Hamilton-Jacobi equations, characterization of various notions of convex functions and sets via first or second order equations, first-order analysis on metric spaces.

４．研究成果

We list the research results below:

1. The key scientific questions are studied in two papers. In the first paper, we prove the equivalence between some known notions of solutions to continuous eikonal equation and more general analogs of the Hamilton-Jacobi equations in complete and rectifiably connected metric spaces. The notions considered are that of curve-based viscosity solutions, slope-based viscosity solutions, and Monge solutions. By using the induced intrinsic metric, we reduce the metric space to a length space

and show the equivalence of these solutions to the associated Dirichlet boundary problem. Without utilizing the boundary data, we also localize our argument and directly prove the equivalence for the definitions of solutions. Regularity of solutions related to the Euclidean semi-concavity is discussed as well. In the second paper, we study the eikonal equation in metric measure spaces, where the inhomogeneous term is allowed to be discontinuous, unbounded and merely p-integrable in the domain with a finite p. Generalizing the notion of Monge solutions in our setting, we establish uniqueness and existence results for the associated Dirichlet boundary problem. The key in our approach is to adopt a new metric, based on the optimal control interpretation, which integrates the discontinuous term and converts the eikonal equation to a standard continuous form. We also discuss the Holder continuity of the unique solution with respect to the original metric under regularity assumptions on the space and the inhomogeneous term.

2. The second project is concerned with a game-based interpretation of Hamilton-Jacobi-Isaacs equations in metric spaces. We construct a two-person continuous-time game in a geodesic space and show that the value function, defined by an explicit representation formula, is the unique solution of the Hamilton-Jacobi equation. Our result develops, in a general geometric setting, the classical connection between differential games and the viscosity solutions to possibly nonconvex Hamilton-Jacobi equations.

3. The last project concerns a PDE-based approach to the horizontally quasiconvex (h-quasiconvex for short) functions in the Heisenberg group. In the first paper, we provide a characterization for upper semicontinuous, h-quasiconvex functions in terms of the viscosity subsolution to a first-order nonlocal Hamilton-Jacobi equation. We also construct the corresponding envelope of a continuous function by iterating the nonlocal operator. One important step in our arguments is to prove the uniqueness and existence of viscosity solutions to the Dirichlet boundary problem for the nonlocal Hamilton-Jacobi equation. Applications of our approach to the h-convex hull of a given set in the Heisenberg group are discussed as well. The second paper is concerned with a PDE approach to horizontally quasiconvex (h-quasiconvex) functions in the Heisenberg group based on a nonlinear second order elliptic operator. We discuss sufficient conditions and necessary conditions for upper semicontinuous, h-quasiconvex functions in terms of the viscosity subsolution to the associated elliptic equation. Since the notion of h-quasiconvexity is equivalent to the horizontal convexity (h-convexity) of the function's sublevel sets, we further adopt these conditions to study the h-convexity preserving property for horizontal curvature flow in the Heisenberg group. Under the comparison principle, we show that the curvature flow starting from a star-shaped h-convex set preserves the h-convexity during the evolution.

５．主な発表論文等

〔雑誌論文〕　計4件（うち査読付論文　0件／うち国際共著　0件／うちオープンアクセス　0件）

１．著者名 Kijowski Antoni、Liu Qing、Zhou Xiaodan	４．巻 in press
２．論文標題 Horizontally quasiconvex envelope in the Heisenberg group	５．発行年 2023年
３．雑誌名 Revista Matematica Iberoamericana	６．最初と最後の頁 -
掲載論文のDOI（デジタルオブジェクト識別子） 10.4171/RMI/1417	査読の有無 無
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Qing Liu and Xiaodan Zhou	４．巻 in press
２．論文標題 Differential games and Hamilton-Jacobi-Isaacs equations in metric spaces	５．発行年 2022年
３．雑誌名 Minimax Theory and its Applications	６．最初と最後の頁 -
掲載論文のDOI（デジタルオブジェクト識別子） なし	査読の有無 無
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Liu Qing、Shanmugalingam Nageswari、Zhou Xiaodan	４．巻 272
２．論文標題 Equivalence of solutions of eikonal equation in metric spaces	５．発行年 2021年
３．雑誌名 Journal of Differential Equations	６．最初と最後の頁 979～1014
掲載論文のDOI（デジタルオブジェクト識別子） 10.1016/j.jde.2020.10.018	査読の有無 無
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

１．著者名 Liu Qing、Zhou Xiaodan	４．巻 XXII
２．論文標題 Horizontal convex envelope in the Heisenberg group and applications to sub-elliptic equations	５．発行年 2021年
３．雑誌名 ANNALI SCUOLA NORMALE SUPERIORE - CLASSE DI SCIENZE	６．最初と最後の頁 30～30
掲載論文のDOI（デジタルオブジェクト識別子） 10.2422/2036-2145.201907_001	査読の有無 無
オープンアクセス オープンアクセスではない、又はオープンアクセスが困難	国際共著 －

〔学会発表〕　計12件（うち招待講演　12件／うち国際学会　8件）

1．発表者名
Xiaodan Zhou

2．発表標題
Discontinuous eikonal equations in metric measure spaces

3．学会等名
MATRIX-RIMS Tandem Workshop on Geometric Analysis in Harmonic Analysis and PDE（招待講演）（国際学会）

4．発表年
2022年～2023年

1．発表者名
Xiaodan Zhou

2．発表標題
Eikonal equations on metric measure spaces

3．学会等名
RIMS Women in Mathematics（招待講演）（国際学会）

4．発表年
2022年～2023年

1．発表者名
Xiaodan Zhou

2．発表標題
Horizontally quasiconvex envelope in the Heisenberg group

3．学会等名
The 47th Sapporo Symposium on Partial Differential Equations（招待講演）（国際学会）

4．発表年
2022年～2023年

1．発表者名
Xiaodan Zhou

2．発表標題
Characterization of horizontal quasiconvexity in the Heisenberg group and applications

3．学会等名
PDE and Analysis Seminar, University of Pittsburgh（招待講演）

4．発表年
2022年～2023年

１．発表者名
Xiaodan Zhou

２．発表標題
Horizontal quasiconvex envelope in the Heisenberg group

３．学会等名
Himeji conference of Partial Differential Equation（招待講演）（国際学会）

４．発表年
2021年～2022年

１．発表者名
Xiaodan Zhou

２．発表標題
Horizontally quasiconvex envelope in the Heisenberg group

３．学会等名
Geometric and functional inequalities and applications seminar（招待講演）

４．発表年
2021年～2022年

１．発表者名
Xiaodan Zhou

２．発表標題
Horizontally quasiconvex envelope in the Heisenberg group

３．学会等名
PDE seminar of HKST（招待講演）

４．発表年
2021年～2022年

１．発表者名
Xiaodan Zhou

２．発表標題
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３．学会等名
Partial Differential Equations under Various Metrics（招待講演）（国際学会）

４．発表年
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１．発表者名
Xiaodan Zhou

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Horizontal convex envelope in the Heisenberg group

３．学会等名
Chinese Academy of Sciences seminar（招待講演）

４．発表年
2020年

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Xiaodan Zhou

２．発表標題
Hamilton-Jacobi equations on metric spaces

３．学会等名
OIST Analysis and Partial Differential Equations Summer School（招待講演）（国際学会）

４．発表年
2023年～2024年

１．発表者名
Xiaodan Zhou

２．発表標題
Discontinuous eikonal equations in metric measure spaces

３．学会等名
OIST Workshop Geometric Aspects of Partial Differential Equations（招待講演）（国際学会）

４．発表年
2023年～2024年

１．発表者名
Xiaodan Zhou

２．発表標題
Discontinuous eikonal equations in metric measure spaces

３．学会等名
AMS Spring Eastern Sectional Meeting（招待講演）（国際学会）

４．発表年
2024年～2025年

〔図書〕　計0件

〔産業財産権〕

〔その他〕

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６．研究組織

	氏名 （ローマ字氏名） （研究者番号）	所属研究機関・部局・職 （機関番号）	備考

７．科研費を使用して開催した国際研究集会

〔国際研究集会〕　計0件

８．本研究に関連して実施した国際共同研究の実施状況

共同研究相手国	相手方研究機関