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研究課題名（英文）Generalized Moonshine

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研究成果の概要（和文）：対数幾何を用いたねじれnodal代数曲線上の共形ブロック理論を構築し、モンスター頂点代数のねじれ加群に対する高次結合情報とfusion ruleを発見した。そして、スムースな写像に対するhigher frameの一般理論と無限次元crystalline降下理論を導入した。更に、頂点代数とアーベル絡代数の構成方法を与えた。この構成法を、与えられた加群の族と絡作用素と高次結合情報に対して、モンスター頂点代数のねじれ加群に応用し、有限群が作用する無限次元リー代数の族を得た。
本研究の結果により、一般ムーンシャイン予想の幾つかの場合を解決したことになる。特に全ての素数pに対するpAクラスの証明を与えた。

研究成果の概要（英文）：I have developed the theory of conformal blocks on twisted nodal algebraic curves using logarithmic geometry, and applied this to the discovery of fusion rules and higher associativity data for twisted modules of the monster vertex algebra. For this purpose, I have also introduced general theories of higher frames for smooth morphisms, and infinite dimensional crystalline descent theorems. I gave a method for constructing vertex algebras and abelian intertwining algebras, given a family of modules, intertwining operators, and higher associativity data. I applied this construction to twisted modules of the monster vertex algebra, and produced a family of infinite dimensional Lie algebras with actions of large finite groups. By the results of this research, I have proved many cases of the generalized moonshine conjecture, in particular for the classes pA in the monster for all primes p.

研究分野：数学

キーワード：代数　ムーンシャイン

１．研究開始当初の背景

(1) Monstrous moonshine arose in the 1970s, when McKay noticed that the q^1-coefficient of the modular j-function is 196884, while the smallest faithful complex representation of the largest sporadic simple group, called the monster, has dimension 196883. Computation with more coefficients and representations suggested that this was not a coincidence, but instead strongly suggested the existence of an infinite dimensional graded representation of the monster, whose graded dimension gives the q-expansion of the j-function. Further investigation by Conway and Norton suggested that in addition to the graded dimension, the graded characters of elements of the monster also yield interesting modular functions. In particular, Conway and Norton formulated the Monstrous Moonshine conjecture, asserting that the characters are Hauptmoduln, which are modular functions with strong symmetry properties. The representation in question, called $V^\natural$, was constructed in 1988 by Frenkel, Lepowsky, and Meurman using methods inspired by string theory, and Borcherds used it in 1992 to prove the Monstrous Moonshine conjecture.

Additional computations with other groups by Queen and Norton suggested that this phenomenon was not limited to the monster, but also showed up with other finite simple groups. Norton amassed this numerical evidence into the Generalized Moonshine conjecture, which asserts the existence of a distinguished family of projective representations of centralizers of elements in the monster, such that traces of commuting elements yield either Hauptmoduln or constants. Furthermore, these functions satisfy a compatibility condition with respect to an action of the modular group. Dixon, Ginsparg, and Harvey gave a physical interpretation to the projective representations, as twisted sectors of a conformal field theory, and their realization in algebra is given by twisted modules of the vertex operator algebra $V^\natural$. Thus, we expect the Generalized Moonshine conjecture to follow from a sufficiently strong understanding of the theory of twisted modules.

２．研究の目的

(1) The main goal of this project was to prove the Generalized Moonshine conjecture by a suitable strengthening of the Borcherds-Höhn method. This method involves the construction of infinite dimensional Lie algebras equipped with actions of large finite groups, and deducing the Hauptmodul property from twisted denominator identities.

(2) For the construction of Lie algebras, I needed a way to assemble vertex algebras and abelian intertwining algebras from pieces. I intended to prove a theorem that would take as input a collection of modules or twisted modules, together with fusion rules and higher associativity data, and show the existence of a suitable algebraic structure on the direct sum of the modules.

(3) To describe fusion rules and higher associativity data, I intended to prove facts about the spaces of conformal blocks for twisted modules. In particular, I hoped to place the geometric theory of conformal blocks on a firmer footing, generalizing it from a theory on smooth curves to the setting of curves with mild singularities.

(4) To obtain twisted denominator identities, I needed a way to find root multiplicities. By applying a bosonic string quantization functor, the problem is reduced to a subtle question about cyclic group actions on twisted modules. I hoped to resolve various ambiguous roots of unity by some vector-valued method.

３．研究の方法

(1) For the construction of abelian intertwining algebras, I applied the Eilenberg-Mac Lane cohomology of abelian groups to associativity data. After working out some power series manipulations, the associativity data naturally correspond to group cohomology data, and if they satisfy a skew-symmetry condition, they produce abelian group cohomology classes.

(2) For the determination of fusion rules and higher associativity data, I attempted to generalize the theory of vertex algebras on algebraic curves that was developed by Frenkel and Ben-Zvi, and enhanced to allow twisted modules by Frenkel and Szczesny. For this purpose, I investigated logarithmic geometry, the theory of stacks, and the theory of crystals.

４．研究成果

(1) I have shown that if we are given a collection of modules or twisted modules, parametrized by an abelian group, together with spaces of intertwining operators that satisfy higher associativity conditions, then the direct sum admits the structure of a vertex algebra object in a certain braided tensor category. However, passing from there to an abelian intertwining algebra requires an evenness condition to be satisfied, and evenness is

quite tricky to prove. Similarly, if one wishes to assemble vertex operator algebras as simple current extensions, one finds that this evenness problem is the only obstruction. I have developed several techniques for proving evenness, e.g., transferring the property along orbifolds, so it is not necessary to prove it for every individual vertex operator algebra. In particular, I have shown that all twisted modules for the monster vertex algebra are even.

(2) I have introduced a higher-order relative frame construction in a very general context. I have shown that for any smooth equidimensional Deligne-Mumford morphism of fibered categories over schemes, there is a fibered category of relative frames that is schematic over the source, and is a torsor under the automorphism group of a formal polydisc. This is a special case of a theory of restricted Hom stacks that I have developed.

(3) I have shown that log-crystalline descent is effective for any morphism that admits strict étale local sections. This allows for a logarithmic generalization of Beilinson-Bernstein localization and Gelfand-Kazhdan formal geometry in families. In particular, an equivariant crystal on a higher frame torsor descends to a crystal on the base.

(4) I have developed a theory of conformal blocks on nodal curves, by combining the logarithmic formal geometry from result (3) with Nagatomo-Tsuchiya's study of nodal degeneration in the genus zero case. I showed that sheaves of conformal blocks as described by Frenkel and Ben-Zvi, and orbifold conformal blocks as described by Frenkel and Szczesny, naturally extend to the boundary of moduli space, where the canonical connections extend to log connections (i.e., they may develop logarithmic singularities). In particular, factorization of conformal blocks for a regular vertex operator algebra follows from a study of insertion of log points and normalization.

(5) Applying results (1-4) to twisted modules of the Monster vertex algebra, I constructed a family of infinite dimensional Borcherds-Kac-Moody Lie algebras $\mathfrak{m}_g$, one for each element g of the monster simple group. When g is the identity, then the Lie algebra is the Monster Lie algebra constructed by Borcherds. Each such Lie algebra $\mathfrak{m}_g$ has a natural projective action of the centralizer of g.

(6) For the cases where g is in the conjugacy class pA for p a prime, or either $3C$ or $4B$, then I have identified the root multiplicities of $\mathfrak{m}_g$, and shown that the orbifold trace functions are Hauptmoduln. This completes the Hauptmodul part of the Generalized Moonshine conjecture in these 17 cases.

5．主な発表論文等

〔雑誌論文〕（計２件）

1. S. Carnahan, Monstrous Lie Algebras, RIMS Proceedings: Research on finite groups and their representations, vertex operator algebras, and algebraic combinatorics 2013/01/07～2013/01/10, ed. Y. Takegahara, no. 1872 (2014) 83–93.

2. S. Carnahan, Generalized Moonshine II: Borcherds products, Duke Mathematical Journal, vol. 161 no. 5 (2012) 893–950, 査読有り.

〔学会発表〕（計７件）

1. S. Carnahan, Dong-Li-Mason plus Weil. December 16, 2014, Workshop on finite groups and their representations, vertex operator algebras, and algebraic combinatorics, RIMS, Kyoto University

2. S. Carnahan, Introduction to Mathieu Moonshine. August 25, 2014, Workshop on Mathieu Moonshine, Tambara International Research Center

3. S. Carnahan, Dong-Li-Mason plus Weil. June 27, 2014, Algebras, Groups and Geometries 2014, University of Tokyo

4. S. Carnahan, Dong-Li-Mason plus Weil. March 22, 2014, Hualien Workshop on Finite Groups, VOA, Algebraic Combinatorics, and Related Topics, National Dong Hwa University（台湾）

5. S. Carnahan, Monstrous Lie algebras. August 27, 2013, Conference on Moonshine, Mock modular forms, and String theory, Simons Center for geometry and physics, SUNY Stony Brook（アメリカ）

6. S. Carnahan, Monstrous Lie algebras. March 27, 2013, Taitung Workshop on group theory VOA and algebraic combinatorics, Taitung University（台湾）

7. S. Carnahan, Monstrous Lie algebras. January 8, 2013, Workshop on finite groups and their representations, vertex operator algebras, and algebraic combinatorics, RIMS, Kyoto University

6．研究組織

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