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科学研究費助成事業　　研究成果報告書

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研究課題名（和文）Trace functionals and operator inequalities with applications in quantum information

研究課題名（英文）Trace functionals and operator inequalities with applications in quantum information

研究代表者

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研究成果の概要（和文）：Golden-Thompson不等式とJensen不等式を補間して，Golden-Thompsonトレース不等式の多変数版を発見した．また，２変数のGolden-Thompson不等式を変形された指数関数の場合に拡張した．正則作用素関数のパースペクティブ理論を基に多変数の幾何平均を構成する一般的な方法を提案した．これは多変数の幾何平均として知られた全ての例を包含するとともに，興味ある新しい例も与える．獲得されたデータに適合して更新手順を設定する点に新規性がある．Von Neumannエントロピーを熱力学の２つの基本性質から特徴づけた．正値確率変数の平均の期待値に対する不等式を示した．

研究成果の概要（英文）：We found a multivariate extension of Golden-Thompson's trace inequality, which may be considered as an interpolation inequality between Golden-Thompson's inequality and Jensen's inequality, and we extended the two variable Golden-Thompson inequality to deformed exponentials. We proposed a general procedure to construct multivariate geometric means based on the theory of perspectives of regular operator maps. The method is general enough to encompass all known examples of multivariate geometric means, and it also provides new interesting examples. A novel feature is that we may impose an updating procedure adapted to data acquisition. We characterised the von Neumann entropy as the only possible entropic measure satisfying two fundamental properties coming from thermodynamics. We obtained an inequality for expectations of means of positive random variables.

研究分野：　数理物理学

キーワード：　量子情報　Golden Thompson's ineq.　Quantum information　Statistical mechanics　Quantum entropy　geometric means　Rao's inequality

１．研究開始当初の背景

The principal investigator has a background in non-commutative analysis, in particular operator inequalities, operator convex functions of several variables, trace functions and related areas. These techniques have been ever more important tools in mathematical physics including quantum information theory and quantum statistical mechanics.

２．研究の目的

The purpose of the research is to advance our knowledge of and improve the mathematical techniques in quantum information theory and related areas. The mathematical investigations are within the theory of operator inequalities, convexity or monotonicity of operator mappings, trace functionals, and entropy functionals.

３．研究の方法

Theoretical investigations.

４．研究成果 The research carried out under the kakenhi funding may be divided into a number of distinct parts that all fall within the main purpose of the program as stated in the application.

1. Inequalities related to Golden-Thompson's trace inequality. We extended this inequality, which is important in quantum statistical mechanics, in two separate directions. We first found a multivariate extension that may be considered an interpolation inequality between Golden-Thompson's inequality and Jensen's inequality. We then extended the two variable Golden-Thompson inequality to deformed exponentials with parameters in the interval [1,3]. We believe this extension will be useful in the theory of non-additive entropies and non-extensive statistical mechanics. Both types of extensions rely on results and techniques in the theory of convex trace functions studied by the author in previous articles.
2. Regular operator mappings. Such mappings share the same regularity properties as the operator function defined by the functional calculus of a real function of one variable. They are also known in the literature as non-commutative functions. We show that convex regular operator mappings satisfy a certain type of Jensen inequality. We defined the perspective of regular operator mappings and proved that they preserve convexity. We finally proved that a convex positively homogeneous regular operator mapping of k+1 variables is the perspective of its restriction to k variables. This result is very useful in the theory of multivariate geometric means.

3. Multivariate geometric means of operators. We proposed a general procedure to construct multivariate geometric means based on the theory of perspectives of regular operator mappings. The method is general enough to encompass all known examples of multivariate geometric means and also provides new interesting examples. The geometric means constructed in this way all generalize the well-known geometric mean of two operators. A novel feature of the theory is that we extend a mean from k to k+1 variables by an updating procedure that may be adapted to a specific problem. We believe this is useful in information processing and data acquisition. We also proved that the Sagae-Tanabe means in this context are uniquely defined by a few natural assumptions.
4. After much revision of earlier preprints we obtained final publication of our work about the characterization of matrix entropies, in cooperation with Zhihua Zhang.
5. In addition to the completion of earlier work we opened up research in two new areas both connected to quantum information theory. We connected the earlier developed theory of regular operator mappings with the theory of partial traces and more generally completely positive mappings and found many new applications in quantum information theory and quantum physics. We also reconsidered the von Neumann entropy and found a surprisingly simple proof of an important classical result by connecting it with the earlier developed theory of matrix entropies.
6. We discovered and proved that the von Neumann entropy up to normalisation is the only measure of quantum entropy satisfying two fundamental requirements coming from thermodynamics. This clarifies a long-standing problem in quantum physics. If there were other substantial different ways of defining quantum entropy, then it could happen that some properties derived for a specific physical system were mere mathematical artefacts of the chosen entropy function. They would be in accordance with the underlying physical principles from thermodynamics, but they would also reflect an arbitrary mathematical choice. We now know that this cannot happen. The von Neumann entropy may increase when passing to a subsystem. This is called the intuitive defect in quantum physics. Our recent result implies that this defect cannot be remedied by possibly adopting an alternative definition of quantum entropy.
7. We discovered an inequality for expectations of means of positive random variables, in cooperation with P. Gibilisco. It

is a vast generalization of Rao's inequality.

５．主な発表論文等

〔雑誌論文〕（計 10 件）

The research results described above are reported in the following papers:

1. Frank Hansen. Perspectives and completely positive maps. Annals of Functional Analysis (2017) **8**(2), 168-176 (peer reviewed).
2. Paolo Gibilisco and Frank Hansen. An inequality for expectations of means of positive random variables. Annals of Functional Analysis (2017) **8(**1), 142-151 (peer reviewed).
3. Frank Hansen. Quantum entropy derived from first principles. Journal of Statistical Physics (2016) **165**(5), 799-808 (peer reviewed). Doi 10.1007/s10955-016- 1651-4.
4. Frank Hansen. A note on quantum entropy. Mathematical Physics, Analysis and Geometry (2016) **19**(7), 1-4 (peer reviewed). Doi 10.1007/ s11040-016-9213-1.
5. Frank Hansen and Zhihua Zhang. Characterisation of matrix entropies. Letters in Mathematical Physics (2015) **105**, 1399-1411 (peer reviewed). Doi 10.1007/s11005-015-0784-8.
6. Frank Hansen. Multivariate extensions of the Golden-Thompson inequality. Annals of Functional Analysis (2015) **6**(4), 301-310 (peer reviewed). Doi 10.15352/ afa/06-4-301.
7. Frank Hansen. Golden-Thompson's inequality for deformed exponentials. Journal of Statistical Physics (2015) **159**(6), 1300-1305 (peer reviewed).
8. Frank Hansen. Regular operator mappings and multivariate geometric means. Linear Algebra and its Applications (2014) **461**, 123-138 (peer reviewed).

Two other papers describing earlier research was published during the current period (2014-2017) of kakenhi funding:

9. Frank Hansen and Edward Effros. Non-commutative perspectives. Annals of Functional Analysis (2014) **5**(2), 74-79 (peer reviewed).
10. Frank Hansen. Trace functions with applications in quantum physics. Journal of Statistical Physics (2014) **154**(3), 807-818 (peer reviewed). Doi 10.1007/ s10955-013-0890-x.

〔学会発表〕（計 7 件）

1. Frank Hansen. Regular operator functions and multivariate geometric means. Invited speaker at the International Linear Algebra Society (ILAS), Seoul Korea, August 6-9, 2014.
2. Frank Hansen. Regular operator functions and multivariate geometric means. Invited speaker at the International Conference on the theory of operator means and related

topics, RIMS Kyoto University, November 26-28, 2014.
3. Frank Hansen. Two contributed talks, 1. Golden-Thompson's inequality for deformed exponentials, and 2. Characterisation of matrix entropies. Invited speaker at the International Congress of Mathematical Physics, Santiago, Chile, July 27-August 1, 2015.
4. Frank Hansen. Inequalitied for quantum channels. Invited speaker at the international conference on Mathematical Aspects in Current Quantum Information Theory (MAQIT), Daejeon, Korea, February 14-19, 2016.
5. Frank Hansen. Quantum entropy derived from first principles. Invited speaker at the international conference on Information Theory and its Applications (IGAIA IV), Liblice, Czech Republic, June 13-17, 2016.
6. Frank Hansen. Matrix inequalities and operator means of several variables. Invited contributed speaker at the 20th conference of the International Linear Algebra Society in Leuven, Belgium, July 11-15, 2016.
7. Frank Hansen. Quantum entropy derived from first principles. Invited speaker at the research conference of geometric structures in quantum information based on operator theory and related topics, RIMS Kyoto University, November 9-11, 2016.

〔図書〕(計　0　件)
〔産業財産権〕

○出願状況（計　0　件）

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〔その他〕
ホームページ等

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