3版

様　式　Ｃ－１９、Ｆ－１９、Ｚ－１９　（共通）

科学研究費助成事業　　研究成果報告書

平成　２８　年　　６　月　２４　日現在

機関番号：１４３０１
研究種目：若手研究(B)
研究期間：2014～2015
課題番号：２６８００００４
研究課題名（和文）Positive Representations

研究課題名（英文）Positive Representations

研究代表者

ＩＰ　Ｉｖａｎ（Ip, Ivan）

京都大学・大学院理学研究科・助教

研究者番号：５０６４６０３１

交付決定額（研究期間全体）：（直接経費）　　2,200,000円

研究成果の概要（和文）：Positive Representationに関する研究を行いました。Borel部分への制限のテンソル積を解明して、量子Teichmuller理論の変異作用素を作りました。なお、正Casimirの固有値を計算し、判別式多様体との関係を研究しました。また、A. Zeitlinとの共同研究によって、ある超代数のQ作用素や、Teichmuller超空間を計算しました。最後に、M.Yamazakiとの共同研究によって、量子Dilogarithmの恒等式の1の冪根への極限が一般化しました。

研究成果の概要（英文）：We have showed that the restriction of positive representation to the Borel part is independent of the parameters λ and also closed under taking tensor product. We construct the quantum mutation operator as part of the candidate of the quantum higher Teichmuller theory. We studied the eigenvalues of the positive Casimir, and its relation to discriminant variety. We develop the idea of virtual highest weight and show that the Clebsch-Gordan coefficients for tensor product decomposition of Uq(sl(2,R)) are certain analytic continuation of the classical coefficients.
In joint works with A. Zeitlin, we studied the Baxter Q operator for Cq(2)(2) and calculate explicitly the R operators using the spinor trick developed for split real case. We also studied the generalization of decorated super Teichmuller space for N=2 with structure group OSp(2|2) Finally with M. Yamazaki, we generalize Bazhanov's quantum dilogarithm identities in the root of unity limit.

研究分野：Representation Theory

キーワード：Positive Representation　quantum groups　modular double　Teichmuller theory　braided tensor category

１．研究開始当初の背景

Finite dimensional representation theory of quantum groups introduced by Drinfeld and Jimbo in 1985, originally motivated for the search of the solutions to the Yang-Baxter equation, has led to many important applications in mathematics and physics over 30 years. In particular the braided tensor category structure gives rise to various 3-manifold and knot invariants, including the famous Reshetikhin-Turaev's TQFT. The finite dimensional representation theory corresponds to one of the real form called the *compact form* in Lie theory. On the other hand, the representation theory of another important real form, the *split real form*, and its corresponding quantization is much less understood.

The notion of ***positive representations*** was initiated in a joint work with I. Frenkel (2012) as a new research program devoted to the study of representation theory of split real quantum groups. In the simplest case of $U_q(sl(2,\boldsymbol{R}))$ studied by Teschner *et al.*, it demonstrates a lot of similarities with the finite dimensional representation theory of the compact case. Most importantly, it shows a braided tensor category structure in a continuous sense, and I have established previously (Ip, 2012) a Peter-Weyl type theorem relating positive quantized functions $L^2(SL_q^+(\boldsymbol{R}))$ with the regular representations of $U_q(sl(2,\boldsymbol{R}))$. However, at the same time it also provides many new phenomena such as Faddeev's modular double, a simple analytic relation giving the Langland's duality, positivity properties and their relations to cluster algebra, and the use of quantum dilogarithm function that is not available in the compact case.

We continue to study this special class of representations and develop the theory further, with the main focus on the braided tensor category structure of the positive representations and its applications. In particular, the study of the tensor product decomposition motivates two directions. First, restriction to the Borel part leads to a new candidate for the quantum higher Teichmüller theory. Secondly, the study of positive Casimir and central characters, which establishes new ideas of *virtual highest weight* which suggests that the positive representation may be related to the finite dimensional representation theory of compact quantum group by certain twisted analytic continuation. As for applications, we will explore the situation when the quantum parameter q goes to root of unity, as well as the specification to the theory of superalgebra.

２．研究の目的

In this project, we develop further the theory of positive representations of split real quantum groups. The aim of the project is to try to establish the tensor product decomposition structure of positive representations, and develop its applications to other field of mathematics. More precisely, we

a) Generalize representation theoretic results of compact quantum groups to the split real case and construct the braided tensor category structure of the positive representations by studying the restriction of positive representations to the Borel part, as well as studying the positive Casimirs and the central characters;

b) Understand the relations between positive representations and quantum higher Teichmüller theory by generalizing Frenkel-Kim's construction for the quantum plane to the Borel part of the split real quantum groups.

c) Develop the properties for the theory of the quantum Lie superalgebra in the split real case as well as the root of unity case.

３．研究の方法

I have been travelling to conferences and workshops in Mathematics (Representation theory) and Physics (Integrable Systems) to give seminars and promote the idea of positive representations of split real quantum groups and its applications, and gave a series of summer school lectures in 2014 at University of Hamburg, Germany on the relation between positive representations and quantum higher Teichmüller theory. New ideas relating C^*-algebra, topological quantum field theory and categorification has been discussed during the conferences with fellow participants, which will prove useful for the new applications of the positive representations. Future directions have also been discussed with Igor Frenkel from Yale University to develop further the theory with emphasis on the semi-classical limit and root of unity limit of the positive representations.

In summary, the proposed method for aims of the project outlined in the previous section are as followed:

a) We study the tensor category structure of the positive representations by two main approaches.

First, we study the restriction of the positive representations to the Borel part and show that it is closed under taking tensor product. This involves the technique from previous work on multiplier Hopf algebra where we construct explicitly the multiplicative unitary which gives a C* theoretic construction of the tensor product decomposition of the Borel part.

Next, recall that the $U_q(sl(2,\mathbf{R}))$ tensor product decomposition involve the study of the spectral decomposition of the central Casimir operator, hence in this project we initiated the study of the central elements in higher rank. We define the positive Casimirs and study its action and eigenvalues of central characters on the positive representations. We expect the spectral decomposition of the Casimir will provide the required intertwiners for the tensor product decomposition.

b) We study the quantum higher Teichmüller theory by generalizing Frenkel-Kim's construction using restriction of the positive representations to the Borel part. In Frenkel-Kim's construction, an important ingredient called the *quantum mutation operator* is constructed by decomposing the tensor product of two representations of quantum plane into tensor product of the same representations with a trivial module. We observed the same phenomenon for the restriction of the positive representations to the Borel part. Another ingredient is the Kashaev's **A** operator, which is constructed from the relation of the representations with its dual associated with the antipode. We propose to construct Kashaev's **A** operator again by the restriction to Borel part, and try to show that it gives Kashaev's groupoid, which will induce a new projective unitary representation of the mapping class and give a new candidate of the quantum higher Teichmüller theory.

c) For the applications to other areas, we study the Baxter's Q operator of the super algebra $C_q^{(2)}(2)$ using the spinor trick proposed for the $U_q(osp(1|2,\mathbf{R}))$ which brings the modular double structure to the super case. We also generalize Penner-Zeitlin's decorated super Teichmüller space to $N=2$ with structure group $OSp(2|2)$ by constructing the group orbits on the light cone explicitly in terms of certain bosonic and fermionic parameters. Finally, we also develop the quantum dilogarithm identities at the root of unity limit generalizing Bazhanov's result, which is essential for the study of root of unity limit of the positive representations in the future. Especially in the case of $q\to-1$ it gives a new classical representation of the quantum plane and provides the first step towards the study of the semi-classical limit of the positive representations.

４．研究成果

a) We studied the restriction of the positive representations to the Borel part of the split real quantum group. We have showed that the restriction of positive representation to the Borel part is canonical and does not depend on the parameters λ. Previously we defined the C* algebraic version of the quantum group by observing that the Lusztig's isomorphism can be casted into conjugation by the Weyl elements, hence all the root generators can be realized by positive operators. This gives the notion of a continuous PBW basis, and defines a C* algebra of bounded operators. In particular we can consider the natural GNS representation of such C* algebra. In (Ip 2014), we show that this representation is unitary equivalent to the restriction of the canonical positive representations. In particular, the GNS representation equips with the multiplicative unitary, which gives us a closed formula for the closure under tensor product.

We have also constructed and studied the eigenvalues of the positive Casimir. Here we introduced a new idea called the *virtual highest weights vectors* such that they are formally the eigendistributions of the generators K, and annihilated by the the generators E. Using this technique, we can follow the compact case and calculate the eigenvalues by looking into the quantum trace of the Cartan part of the product of the universal R matrix. We found that the central characters are given by positive scalars on the positive representations, which are bounded below by the dimensions of the fundamental representations of the *compact quantum groups*. This gives an important link between the compact and split real cases.

By considering the image of all the eigenvalues, they bounded a semi-algebraic region such that the boundary is given by certain discriminant variety which is studied in primitive forms and singular theory. The precise relation is yet to be researched.

Finally we show that the Clebsch-Gordan coefficients for tensor product decomposition of $U_q(sl(2,\boldsymbol{R}))$ obtained by unitary transformations by the quantum dilogarithms are certain analytic continuation of the classical coefficients, using the above idea of virtual highest weight developed for the calculation of the positive Casimir. This suggests that we should define precisely certain twisted analytic continuation as the correct way to study the link between the results of compact and split real representation theory.

b) As a result of the tensor product decomposition of the restriction of positive representations to the Borel part, it provides us a quantum mutation operator, generalizing the similar construction by Frenkel-Kim in the case of quantum plane. This gives a candidate of the quantum higher Teichmüller theory. However, in an ongoing work with H. Kim, we have not yet successfully found the Kashaev's **A** operator in the higher rank case. By studying the dual representation, we naturally define an **A** operator satisfying $\mathbf{A}^3=1$, however it is not compatible with the quantum mutation operator, in particular the reflection equation is not satisfied. This problem will remain to be done in a future project.

c) In joint works with A. Zeitlin, we studied the Baxter Q operator for $C_q^{(2)}(2)$ and calculate explicitly the R operators again using the spinor trick which relates the representation of super Lie algebra $U_q(osp(1|2,\boldsymbol{R}))$ with those of the split real form $U_q(sl(2,\boldsymbol{R}))$ given by evaluation representations. We also studied the generalization of decorated super Teichmüller space for $N=2$ with structure group $OSp(2|2)$, generalizing the construction by Penner-Zeitlin for the case of $N=1$. This provides a first step in the construction of quantum higher Teichmüller theory in the super case, and the positive representations will play an important role just as in the classical situation, with the quantum dilogarithm giving the mutation operators of the super coordinates. As of the end of FY2015 we are still preparing the submission of the manuscript.

Finally in a joint work with M. Yamazaki, we generalized Bazhanov's quantum dilogarithm identities in the root of unity limit, by introducing the idea of modular double into cluster variables, and also using the explicit root of unity limit of the quantum dilogarithm discovered recently by Kashaev *et al.* This will provide new clues to the root of unity limit of the representation theory of quantum plane, which is the building block of the more general positive representations of split real quantum groups.

５．主な発表論文等
（研究代表者、研究分担者及び連携研究者には下線）

〔雑誌論文〕（計　6　件）

1) <u>I. Ip</u>, A. Zeitlin,
Supersymmetry and the modular double, Contemporary Mathematics, Vol 623, 81-97, (2014) (refereed)

2) <u>I. Ip</u>, A. Zeitlin,
Q-operator and fusion relations for $C_q^{(2)}(2)$*,* Letters in Mathematical Physics, Vol 104 (8), 1019-1043, (2014) (refereed)

3) <u>I. Ip</u>, M. Yamazaki,
Quantum dilogarithm identities at root of unity, International Mathematics Research Notices, Vol 2016 (3), 669-695, (2016) (refereed)

4) <u>I. Ip</u>,
Positive Casimir and central characters of split real quantum groups, Communication in Mathematical Physics (to appear) (2016) (refereed)

5) <u>I. Ip</u>,
On tensor products of positive representations of split real quantum Borel subalgebra $U_{qq'}(b_{\boldsymbol{R}})$*, arXiv:1405:4786 (preprint)*

6) <u>I. Ip</u>,
On tensor products decomposition of positive representations of

decomposition of $U_{qq'}(sl(2,\mathbf{R}))$*,*
arXiv:1511.07970 (preprint)

〔学会発表〕（計　8　件）

1. *“Positive Representations and Quantum Higher Teichmüller Theory”*, International Conference "Mathematics Days in Sofia", Bulgaria (2014 July)

2. *“Positive Representations and Quantum Higher Teichmüller Theory”*, The 30th International Colloquium on Group-Theoretical Methods in Physics, Ghent University, Belgium (2014 July)

3. *“Positive Representations and Quantum Higher Teichmüller Theory”*, Summer School on Quantum Groups and Integrability, University of Hamburg, Germany (2014 July, 3 lectures)

4. *“Positive Representations and Quantum Higher Teichmüller Theory”*, RIMS Project 2014 on Geometric Representation Theory, RIMS, Kyoto (2014 July)

5. “*Quantum dilogarithm identities at root of unity*”, Seminar on Pure Mathematics, HKUST, Hong Kong (2015 July)

6. “*Quantum dilogarithm identities at root of unity*”, Geometry Seminar, HKU, Hong Kong (2015 Sept)

7. “*Positive Casimir and Central Characters of Split Real Quantum Groups”*, Geometry, Symmetry and Physics Seminar, Yale University, USA (2016 Mar)

8. “*Positive Casimir and Central Characters of Split Real Quantum Groups”*, Columbia Symplectic Geometry, Gauge Theory and Categorification Seminar, Columbia University, USA (2016 Mar)

〔図書〕（計 0　件）

〔産業財産権〕
○出願状況（計 0　件）

名称：
発明者：
権利者：
種類：
番号：
出願年月日：
国内外の別：

○取得状況（計 0　件）

名称：
発明者：
権利者：
種類：
番号：
取得年月日：
国内外の別：

〔その他〕
ホームページ等
N/A

６．研究組織

(1)研究代表者
IP Ivan（Ip, Ivan）
京都大学・大学院理学研究科・助教
研究者番号：５０６４６０３１

(2)研究分担者
N/A

(3)連携研究者
N/A