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研究成果の概要（和文）：腫瘍や結晶がどのように成長していくかのような自然科学における現象で、それが成長していく際の境界がどのように振る舞うかは、自由境界問題で記述されます。このような問題に、比較原理の性質に基づく粘性解の概念を用いて解析を行いました。均質化法を活用し、氷の性質における小規模な変化が溶融の速度にどのように影響するかを数学的に示しました。また、腫瘍成長および個体群動態論の境界がはっきりしたものと、グラデーションになっているものとの間の関係を解明しました。最後に、多次元結晶成長モデル（クリスタライン平均曲率流）の粘性解の新たな概念を導入し、それらの存在性、一意性および安定性を証明しました。

研究成果の概要（英文）：Many problems in applied sciences, for example the growth of tumors or crystals, are described by nonlinear differential equations with a moving interface. We worked on the analysis of such problems using the notion of viscosity solutions that rely on a order-preserving property of solutions (maximum principle) in these problems. We showed how a small-scale variations in the properties of ice influence the speed of melting using the homogenization approach. We also clarified the relation between sharp interface and diffusive interface models of tumor growth and population dynamics, including situations with a drift field. Finally, we introduced a new notion of viscosity solutions for a model of crystal growth (the crystalline mean curvature flow) in an arbitrary dimension and proved its existence, uniqueness and stability. This opens the possibility for further rigorous study of this model.

研究分野： Mathematical analysis, Applied analysis

キーワード： homogenization　crystal growth　viscosity solutions　Hele-Shaw problem　phase transitions　porous medium equation

１．研究開始当初の背景

Moving interfaces are common in scientific research, including fluid interfaces, surfaces of crystals, boundaries between different tissues such as a tumor and a healthy tissue in biology, the edge of a crowd of people in population dynamics, etc. However, it has been challenging to understand their behavior due to the mathematical difficulties they pose as many quantities abruptly change across them. Not only that, in applications their motion is often subject to highly oscillating influences on a very small scale. In recent years, there has been a significant interest among mathematicians to develop tools to help with the understanding of the various phenomena in which moving interfaces appear.

２．研究の目的

We set to further develop the techniques in the theory of generalized solutions called viscosity solutions to investigate problems with a moving interface that have highly oscillating coefficients. We focused on the Hele-Shaw problem and a motion driven by the mean curvature. To understand the large scale influence of oscillating coefficients, we used the homogenization approach. We also sought to use the developed techniques to understand other singular limits of solutions, and to generalize the notion of viscosity solutions to other important problems.

３．研究の方法

This work is based on an individual research as well as joint work with overseas and local mathematicians. To facilitate these collaborations and to disseminate the obtained results, I participated in numerous conferences and short research visits of my collaborators, and hosted invited researchers. The main theoretical work in mathematical analysis was supported by performing various numerical computations using algorithms that were themselves motivated by the theoretical results.

４．研究成果

We have made contributions in the following three directions ([ ] below refer to published papers in Section 5):

(1) Homogenization of free boundary problems of a Hele-Shaw type

The Hele-Shaw is a popular model of the flow of liquids through a porous medium like sand and soil or the flow through narrow spaces such as the injection molds in the plastic industry. The moving interface appears between the flowing liquid and gas. Its shape might be quite complicated and events like collision and splitting of its parts create singularities that are challenging to handle mathematically. The pressure that drives the flow is influenced by the overall shape of the wet region and it is therefore a nonlocal problem. Generalized solutions like viscosity solutions that rely on an order-preserving property (the maximum principle) are necessary to even define a solution. In the early work [7] relying on the maximum principle, we showed that even if the moving interface advances with a velocity that is influenced by a coefficient that oscillates both in space and time, on a much larger scale it behaves like a solution with a coefficient that only depends on the direction. The main contribution here was obtaining a novel bound on the local oscillations of the moving boundary at large scales. Paper [2] deals with a similar setting for the related Stefan problem that models a phase transition as ice melts into water by a localized heat source. Here we show that even if the ice has small scale variations of the latent heat of phase transition in space, over long time this is averaged out and only the mean latent heat influences the asymptotic speed. We combine the viscosity solutions with the variational structure of this problem and show how to handle a localized heat source.

(2) Incompressible limit of the porous medium equation (PME) is the Hele-Shaw problem

In cell biology and population dynamics, for example models of tumor growth, there are two popular approaches to the modeling: sharp interface models that split the domain into two or more sharply separated regions like tumor/healthy tissue, often a Hele-Shaw type problem explained above, and diffuse interface models that treat the transition more gradually, for example the PME, which treats the tumor as a compressible material that mixes with the healthy tissue. However, their exact relationship has not been rigorously established in this context. Using the viscosity solution techniques, in particular a methodology reminiscent of the maximum principle proof used here to carefully analyze singular limits of ordered solutions, we succeeded in giving a precise characterization of the relationship in [1,4] in that the Hele-Shaw problem is the incompressible limit of the PME. In particular, when a drift field carries the tumor cells through the domain, the limit problem is no longer monotone in time. We overcame this difficulty in [1] by taking advantage of a monotonicity along the streamlines of the flow.

(3) Notion of viscosity solutions for the crystalline mean curvature flow

Crystals of a solid substance are characterized by their fascinating shapes with a surface exhibiting flat parts (facets), sharp edges and corners that are preserved as the crystal grows. It is not clear how they could be modeled by a differential equation since these surfaces cannot be globally described by a differentiable function. It turns out that we can try to understand their evolution as driven by a surface energy that is strongly anisotropic due to the arrangement of atoms in a crystalline lattice, as observed in the work of Taylor (1978). Formally, the gradient flow of this energy leads to a differential-like equation with a very strong singularity and nonlocal behavior. Unfortunately, its solutions cannot be defined in the classical way. Despite that, there had been a lot of successful work in two dimensions on addressing this problem. In our work [3,5,6], we introduced the first general notion of viscosity solutions for this problem in any dimension, established the existence of a unique solution and thus gave a satisfactory answer to a long-standing open problem. We modified the standard theory by allowing for faceted test functions and showed how to modify the proofs of the maximum principle and stability of solutions, without relying on the smoothness of test functions. This work opens a way for a further understanding of the crystal growth phenomena and many possibilities for future research.

５．主な発表論文等

（研究代表者、研究分担者及び連携研究者には下線）

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〔図書〕（計　0 件）

６．研究組織

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